Real Yields and the Transmission of Central Bank Balance-Sheet Policies

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PRELIMINARY DRAFT

Abstract

I show that, while a lower bound on the short-term policy rate limits the response of *nominal* yields to central bank bond purchases, *real* yields, which are more relevant for the macroeconomy, do not face such a constraint. I then study the potential for policies like quantitative easing (QE) to affect real yields and macroeconomic variables using an equilibrium, no-arbitrage framework where investors care about both inflation and duration risk. As long as the policy rate can be expected to respond aggressively to inflation, buying nominal bonds has a larger effect on real yields than buying real bonds. Thus, under normal circumstances, nominal QE is a more efficient way of providing stimulus than real QE. But the effects of nominal QE diminish as nominal rates decline; indeed, a quantitative version of the model suggests that nominal QE purchases were less effective in 2020 than in 2009. In a future recession where $r*$ has fallen even further, nominal QE could stop working altogether, while central banks could still provide stimulus by buying real bonds—or, equivalently, by lending to financial institutions at an inflation-indexed rate.

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The views expressed do not represent those of the Federal Reserve Bank of Chicago or the Federal Reserve System.

1 Introduction

With short-term policy rates reaching their lower bounds in recent years, central banks have turned to large-scale purchases of government bonds—colloquially known as "quantitative easing" (QE)—to provide monetary stimulus. Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2012), D'Amico and King (2013), and many other empirical studies have shown the effectiveness of QE in lowering nominal bond yields so far. Yet, because long-term nominal yields themselves cannot fall substantially below the same lower bound that applies to short rates, their responsiveness to QE must at some point diminish. King (2019) and Gagnon and Jeanne (2020) demonstrate this result in particular structural models, but it is straightforward to show that it holds in a broad class of models where arbitrage opportunities are ruled out.

The persistent downward trend in yields in developed economies over the last several decades, depicted in Figure 1, is thus a source of concern for policymakers who might otherwise favor QE policies. The ten-year nominal U.S. Treasury rate was 3.61% in March 2009, the day before the Federal Reserve first announced Treasury purchases in its "QE1" program; by the time the Fed revived its bond buying in response to the Covid-19 pandemic eleven years later, that yield stood at just 1.03%.[1] A nominal yield of barely one percent, relative to a lower bound of approximately zero percent, seems to leave limited space for QE to do its work. In some jurisdictions the apparent room to maneuver has been even smaller, and this perception has partially motivated a shift into more-exotic forms of unconventional monetary policy, such as negative policy rates and purchases of private assets.[2]

Yet it is worth noting that the lower bound on the term structure only applies to *nominal* yields. Long-term *real* yields can in principle become arbitrarily negative because short-term *real* rates are not subject to a floor. In most theoretical macroeconomic models it is real rates of return, not nominal ones, that matter for economic activity. Empirically, Abrahams et al (2016) show that the real term premium is the component of the yield curve that has responded most strongly to QE, while Gilchrist et al. (2015) show that shocks to the real term premium resulting from unconventional monetary policy pass through to private borrowing costs. Meanwhile, Gertler and Karadi (2015) suggest that transmission through long-term real rates accounts for

[1] Gurkaynak et al. (2007) zero-coupon yields on March 16, 2009, and March 19, 2020. Data maintained at federalreserve.gov.

[2] See, e.g., Bartsch et al. (2019); Carlson et al. (2019); Reifschneider and Wilcox, (2020); Rosengren (2019).

a substantial portion of conventional monetary-policy shocks on the real economy.[3]

Together, these observations seem to hint that QE could still provide economic stimulus when long-term nominal yields are constrained if it remains effective in reducing real term premia in such an environment. I study this possibility in an equilibrium model of real and nominal debt. I show that a central bank can still affect real yields when the nominal yield curve is flat at its lower bound, but only by buying inflation-linked bonds or engaging in other, economically equivalent, transactions.

In the model, when the central bank follows the Taylor Principle in setting short rates, purchases of nominal bonds actually have *greater* effects on real yields than purchases of real bonds do. This is because nominal bonds involve more real interest-rate risk than real bonds.[4] But the covariance between the two types of bonds, and thus the effect of nominal QE on real yields, decreases as nominal yields approach the effective lower bound (ELB). At some point, the covariance drops low enough that buying nominal bonds becomes less effective than buying real bonds. In the limit, when the ELB binds at long maturities, nominal QE ceases to have any effect at all. But, even in that state, purchases of real bonds can still push real yields to arbitrarily low values. One way to think about the latter result is that, by buying real bonds, the central bank removes a good inflation hedge from investors' portfolios and thereby raises the compensation investors require for bearing inflation risk. When nominal yields are stuck at their lower bound, the inflation-risk premium can only rise if real yields fall.

To examine the effects of alternative QE policies on yields and the economy in different states of the world, I consider a quantitative, macro-finance version of the model in which the consolidated government issues arbitrary quantities of real and nominal debt, which risk-averse investors must hold in equilibrium, across a continuum of maturities. QE operations can be viewed as exogenous reductions in these bond quantities. The short-term nominal rate in the model follows an inertial Taylor rule, subject to random deviations and restricted by an effective lower bound. Meanwhile, inflation and real output depend stochastically on the level of longer-term real yields. The model is paramaterized using data on real and nominal bond yields, as well as

[3] In recent work, Hanson et al. (2021) argue that the real term premium effects of conventional monetary policy shocks, of the type documented in Gertler and Karadi (2015), are quite transitory. But their paper does not speak to the effects of QE shocks or the extent of transmission conditional on such a shock occurring.

[4] The result is similar to one pointed out by Diez de los Rios (2020), although in his model real debt is all short-term and thus involves no duration risk.

GDP and inflation, in the U.S. over the last 20 years.

When the model is tuned to an environment similar to that prevailing at the beginning of QE1, an amount of nominal bond purchases that lowers the 10-year nominal yield by 100 basis points, holding the expected short rate fixed, lowers the 10-year real yield by about 86 basis points—a response roughly consistent with empirical observation. The lower real yield translates into an increase in inflation of about 0.3% and a cumulative increase in output of about 0.6% over the next five years. If the same amount of purchases were instead directed to *real* bonds, the model implies that the real yield would fall by just 58 bp, with correspondingly lower macroeconomic effects. In an environment similar to that prevailing at the beginning of the Covid-19 crisis, with the ELB closer to binding on nominal yields, the effects of QE on real yields and the economy are reduced by about one-third, although nominal QE continues to be more effective at reducing real rates than real QE. But in a hypothetical future recession in which the equilibrium real short rate ($r*$) has fallen below -1%, the nominal yield curve becomes flat at zero, and nominal QE no longer has any effect on yields or the economy. In this environment, buying real bonds continues to reduce real yields and indeed is still about as effective as in the higher-rate scenarios. From these results, I conclude that purchases of long-term inflation-linked debt could be a powerful tool in reducing real rates and stimulating economic activity during future ELB episodes.

A practical difficulty with this policy, however, is that inflation-linked bond markets are small compared to the overall size of government bond markets. Even where available, central banks have generally shied away from buying inflation-linked debt out of a concern that a disproportionate footprint in that market could interfere with liquidity and functioning. For example, indexed bonds in the U.S. (TIPS) are only about 5% of total Treasury debt outstanding and have constituted less than 3% of the Federal Reserve's bond portfolio since it began its Treasury QE program in 2009. Indeed, the modest market size across economies may seem to place a hard limit on the the quantity of real duration that a central bank can absorb from private agents.

But there is a workaround. Instead of buying real bonds, central banks could provide term credit to financial institutions at an inflation-indexed rate. The inflation exposure that this would transfer from the government to the private sector would create the same changes in nominal risk positions as QE purchases of inflation-linked bonds, thereby increasing the premium for inflation risk and driving down real yields. Moreover, lending at negative real long-term rates would avoid a number of the objections that have been raised about central banks attempting to engineer negative nominal short rates. Under current law, the Federal Reserve requires the invocation of its emergency 13(3) powers (including consent of the Treasury) to undertake term lending of nearly any kind, so the hurdles to implementing this type of program would be relatively high. On the other hand, other central banks, such as the ECB, are already practiced in providing term credit and would seem to require only modest operational changes to switch to inflation indexation.

This paper fits within the growing literature studying fluctuations in bond supply in equilibrium no-arbitrage yield-curve models, following Vayanos and Vila (2021) (e.g., Hamilton and Wu, 2012; Greenwood and Vayanos, 2014; Greenwood et al., 2018; Hanson et al., 2021). King (2019) incorporated the ELB into a model of this type, similarly to this paper, but, as most other work in this literature, ignored inflation. This is an important omission, not just because of the policy issues that are my focus here, but because inflation risk generally is a key factor in bond pricing (see Gurkaynak and Wright, 2012). Consequently, its absence has made these models difficult to reconcile with the rest of the term structure literature, to use for quantitative analysis, and to think about the distinctions between real and nominal bonds.[5] The only paper in this vein to explicitly incorporate inflation is Diez de los Rios (2020), which shares some similarities with the analysis here. However, that model assumes away price rigidity, the nominal lower bound, and long-term real bonds, so it is not equipped to address the policy questions I study. The present paper is also among the first to incorporate real activity into Vayanos-Vila-style models of the yield curve. The macroeconomic block of my model is related to Ray (2019), who introduces a Vayanos-Vila mechanism into a linear New Keynesian model with long-term nominal bonds.

Empirically, this paper relates to the literature, some of which was mentioned above, studying the observed impact of unconventional monetary policy on the term structure and other variables. As noted by Carlson et al. (2020), there is so far little empirical work testing the "diminishing returns" to nominal QE. My quantitative results suggest that at least one of the channels of QE has indeed become smaller in the low-rate environment.[6] This paper also helps to rationalize the observed effects of QE

[5] Kaminska and Zinna (2019) estimate a Vayanos-Vila-type using TIPS yields, but their model also abstracts from inflation and the ELB.

[6] D'Amico and Seida (2020) look at time-variation in the "local supply" effects of Federal Reserve balance-sheet actions and find no systematic trend, suggesting that the lower-yield environment has not mattered much. However, local-supply effects are distinct from those that appear in models of the type studied here, where term premia are determined by the duration risk faced by investors when

announcements on nominal versus real yields reported in Krishamurthy and Vissing-Jorgensen (2012) and Abrahams et al. (2016). More broadly, it relates to the literature on the relative pricing of nominal and inflation-indexed debt. Empirical work in this field (e.g., Evans (2002); Ang et al. (2008); D'Amico et al. (2018)) has documented the presence of significant and time-varying inflation-risk premia and real term premia, but the economic determinants of those premia remain somewhat elusive. I show how both premia may be determined, at least in part, by the quantities of real and nominal exposures that investors hold.

In the following section, I review the reason for the lower bound on nominal yields in a broad class of no-arbitrage models and why such a bound does not apply to real yields. In Section 3, I consider a toy model where closed-form solutions are attainable and where quantities of real and nominal bonds matter for term premia, and I use the model to gain some intuition about the effects of different QE policies. Section 4 expands the model to a spectrum of maturities and calibrates the model to the data on bond yields and macro variables. Section 5 runs the quantitative QE-like policy experiments in different environments. Section 6 discusses some implementation considerations, including the possibility that inflation-linked bond purchases could be replaced with inflation-indexed loans to financial institutions. Section 7 concludes.

2 Bond yields and the lower bound under no-arbitrage

Gagnon and Jeanne (2020) observed that any lower bound on short rates also applies to long-term yields in a specific structural model of consumers with heterogeneous investment horizons. This result is actually a general property of models that rule out equilibrium arbitrage opportunities, as I now show. In principle, the bounding argument applies to either real or nominal yields if the real or nominal short rate, respectively, is itself bounded. However, in practice, the nominal short rate is bounded while the real short rate is not, so no-arbitrage conditions are consistent with arbitrarily negative real yields.

2.1 Nominal bonds

Using discrete time, let the time-t price of a zero-coupon bond that returns a nominal payoff of \$1 with certainty n periods in the future be denoted by $P_t^{\$(\tau)}$. Because coupon-

bonds are perfect substitutes and no-arbitrage holds.

bearing bonds can be constructed as portfolios of zero-coupon bonds, consideration of this case is without loss of generality for what follows. The definition of the "yield" on this bond is $y_t^{\$(\tau)} \equiv -\log P_t^{\$(\tau)}/\tau$.

Suppose that the one-period nominal yield $y_t^{\$(1)}$ is bounded from below by the value b in all periods. Then, the bond price $P_t^{\$(1)}$ is bounded from above by the value $B = e^{-b}$ in all periods. This of course means that $E_t\left[P_{t+h}^{\$(1)}\right]$ is also bounded by B at any horizon h, where $E_t[.]$ is the expectation conditional on time-t information, taken under the standard, "physical" measure $\mathbb{P}$. The logic for why *long*-term nominal bond prices are bounded is that those prices are expectations under the risk-neutral measure $\mathbb{Q}$ and zero-probability events under $\mathbb{P}$ are also zero-probability events under $\mathbb{Q}$. Thus, any upper bound under $\mathbb{P}$ must also constitute an upper bound under $\mathbb{Q}$.

To formalize this claim, recall that an absence of equilibrium arbitrage opportunities implies the existence of a one-period real stochastic discount factor $M_{t+1} \geq 0$ that prices all assets in the economy according to

$$P_t^{\$} = \mathrm{E}_t\left[M_{t+1}\frac{P^{\$\prime}_{t+1}}{\Pi_{t+1}}\right] \tag{1}$$

where $P_t^{\$}$ is the price of an arbitrary nominal asset, $P^{\$\prime}_{t+1}$ is its next period nominal value, Π_{t+1} is the gross rate of inflation between periods t and $t+1$. Let s_t be a random vector denoting the state of the economy and following a known Markov process on the Borel set S. Denote the transition probability from an arbitrary state value at time t to another at time $t+1$ as $\omega^{\mathbb{P}}(s_{t+1}|s_t)$, where the $\mathbb{P}$ superscript denotes that these are probabilities under the physical measure. Then we can write the time-t SDF as a function of the time-t and $t-1$ states, $M_t = M(s_{t-1}, s_t)$. Similarly, inflation and bond prices can be written as functions of the time-t state: $\Pi_t = \Pi(s_t)$ and $P_t^{\$(\tau)} = P^{\$(\tau)}(s_t)$. Expectations as of time t are also conditional on the time-t state. For example, we have $E_t[M_{t+1}] = E[M(s_t, s_{t+1})|s_t] = \int_S \omega^{\mathbb{P}}(s_{t+1}|s_t)M(s_t, s_{t+1})ds_{t+1}$.

By (1), the price of a nominal two-period bond is

$$\begin{aligned} P_t^{\$(2)} &= E_t\left[M_{t+1}\frac{P_{t+1}^{\$(1)}}{\Pi_{t+1}}\right] && (2) \\ &= \int_S \omega^{\mathbb{P}}(s_{t+1}|s_t)\frac{M(s_t, s_{t+1})P^{\$(1)}(s_{t+1})}{\Pi(s_{t+1})}ds_{t+1} && (3) \end{aligned}$$

Define the risk-neutral probabilities $\omega^{\mathbb{Q}}$ by the change of measure

$$\omega^{\mathbb{Q}}(s_{t+1}|s_t) \equiv \omega^{\mathbb{P}}(s_{t+1}|s_t)\frac{M(s_t, s_{t+1})}{P^{\$(1)}(s_t)\Pi(s_{t+1})} \tag{4}$$

We then have

$$\begin{aligned} P_t^{\$(2)} &= P_t^{\$(1)}\int_S \omega^{\mathbb{Q}}(s_{t+1}|s_t)P^{\$(1)}(s_{t+1})ds_{t+1} && (5) \\ &\leq P_t^{\$(1)}B && (6) \end{aligned}$$

with equality iff all of the $\mathbb{Q}$ probability mass is on $P_{t+1}^{\$(1)} = B$. This places an upper bound on $P_t^{\$(2)}$, conditional on the current period's short-term bond price. The corresponding lower bound on the two-period yield is

$$y_t^{\$(2)} \geq (y_t^{\$(1)} + b)/2 \tag{7}$$

This lower bound on the yield is conditional, since it depends on the current value of $y_t^{\$(1)}$. Because $y_t^{\$(1)}$ itself is bounded by b in every period, the *unconditional* bound on two-period yields is b—the same bound that applies to the short rate. This argument can be extended inductively to show that nominal yields of any maturity are unconditionally bounded below by b. A stricter conditional bound will generally apply if the time-t short rate is not itself equal to b. In most of what follows, however, we will be concerned with the case where today's short rate is indeed constrained by the ELB, so the unconditional bound is the relevant one.

The economic intuition for why nominal yields are bounded is straightforward. By assumption, we know that the price of a one-period bond will never be greater than B. Consequently, if the price of a two-period bond at time t were greater than $P_t^{\$(1)}B$ it would earn a negative one-period excess return with certainty—an investor would never hold such a bond because she could always do better by investing at the short rate, no matter what state of the world was realized next period. Put differently, risk-free arbitrage profits could be earned by going long the one-period bond and short the two-period bond. Again, this reasoning can be extended to nominal bonds of any maturity. Importantly, it assumes nothing about the determinants of the SDF or any of the specific features of the economic environment. For example, it does not depend on how (or whether) the quantities of Treasury bonds outstanding might matter for asset pricing, as long as assets are valued only for their pecuniary returns and no-arbitrage holds.

Another way of writing the nominal no-arbitrage restriction (6) that will prove useful is to make use of the identity

$$E_t\left[\frac{M_{t+1}}{\Pi_{t+1}}P_{t+1}^{\$(\tau)}\right] = E_t\left[\frac{M_{t+1}}{\Pi_{t+1}}\right]E_t\left[P_{t+1}^{\$(\tau)}\right] + cov_t\left[M_{t+1}, \frac{P_{t+1}^{\$(\tau)}}{\Pi_{t+1}}\right] \tag{8}$$

Substituting (3) and rearranging gives a bound on the conditional covariance between the real payoff of today's τ-maturity bond and the SDF:

$$cov_t\left[M_{t+1}, \frac{P_{t+1}^{\$(\tau-1)}}{\Pi_{t+1}}\right] \leq P_t^{\$(1)}\left(B^{\tau-1} - E_t\left[P_{t+1}^{\$(\tau-1)}\right]\right) \tag{9}$$

for any bond maturity $\tau > 0$. In order for no-arbitrage to hold, the joint distribution of the SDF and bond prices one period ahead must satisfy this condition.

2.2 Real bonds

Note that the above argument does not generally apply to real bonds. As long as inflation is unbounded, short term real rates can take any value with positive probability, so no level of long-term bond prices can ever generate an arbitrage opportunity that is entirely risk-free. Because investors in such bonds are always exposed to some level of risk, it is always possible to find a level of risk aversion, as captured by the SDF, that will rationalize any risk premium and therefore any bond price.

Formally, let $P_t^{(\tau)}$ (with no \$ superscript) indicate the price of a τ-period zero-coupon inflation-indexed bond. Since, by definition, these bonds return the rate of inflation every period in addition to their capital gains, the price of the one-period real bond is just the discounted risk-neutral expectation of next period's inflation:

$$P_t^{(1)} = E_t\left[M_{t+1}\right] \tag{10}$$

$$= P_t^{\$(1)}\int_S \omega^{\mathbb{Q}}(s_{t+1}|s_t)\Pi(s_{t+1})ds_{t+1} \tag{11}$$

$$= P_t^{\$(1)}E_t^{\mathbb{Q}}\left[\Pi_{t+1}\right] \tag{12}$$

where $E_t^{\mathbb{Q}}[.]$ is the time-t expectation operator under the risk-neutral measure. Although it is not essential to any of what follows, it is expositionally convenient to assume that the inflation rate between time t and $t+1$ is known as of time t, and thus

$E_t^{\mathbb{Q}}[\Pi_{t+1}] = \Pi_{t+1}$. (This assumption will be maintained in the two-period model of Section 3 but will be relaxed in the continuous-time model of Section 4.). In this case, taking logs of (12), the Fisher equation holds:

$$y_t^{(1)} = y_t^{\$(1)} - \pi_{t+1} \tag{13}$$

where $\pi_{t+1} \equiv \log \Pi_{t+1}$. It is clear that, regardless of potential restrictions on the nominal rate, the short-term real rate given by this equation can take any real value, as long as inflation is conditionally unbounded.

Meanwhile, the two-period real bond price is

$$\begin{aligned} P_t^{(2)} &= E_t\left[M_{t+1}P_{t+1}^{(1)}\right] && (14)\\ &= P_t^{\$(1)}E_t^{\mathbb{Q}}\left[\Pi_{t+1}P_{t+1}^{(1)}\right] && (15)\\ &\leq P_t^{(1)}BE_t^{\mathbb{Q}}[\Pi_{t+2}] && (16) \end{aligned}$$

where we have again made use of the convenient assumption that $t+1$ inflation is known at time t. Unlike its nominal analogue in equation (6), this "bound" on real yields involves risk-neutral expectations of gross inflation, which, without further restrictions on the SDF or the inflation process, can take any positive value. In particular, if inflation does not have an upper limit, there always exists some SDF that will generate any desired $\mathbb{Q}$ expectation of 2-period inflation, conditional on $P_t^{\$(1)}$, and thus be consistent with any specified level of $P_t^{(2)}$.

Looked at differently, equation (16) can be written in terms of yields as

$$y_t^{(2)} \geq \frac{y_t^{(1)} + (b - E[\pi_{t+2}])}{2} - \frac{1}{2}\log\left[1 + \frac{cov_t[M_{t+1}, \Pi_{t+2}]}{E_t[\Pi_{t+2}]}\right] + J_t \tag{17}$$

where J_t is a Jensen's inequality term that depends on the (physical) distribution of Π_{t+2}. The first term on the right-hand side is the expectations component of yields when the one-period nominal rate is known to be at b with certainty next period (cf. equation (7)). The second term is the term premium that applies to two-period real bonds in that state of the world. Even conditional on current levels of the real short rate and inflation, and holding fixed the stochastic process that determines these variables in future periods, the current two-period yield can still take any real value because nothing restricts the covariance term in equation (17)—an SDF that places high enough weight on high-inflation outcomes will result in an arbitrarily negative real term premium.[7]

Because the class of functions that are admissible as SDFs faces essentially no restrictions other than positivity, it is straightforward to find trivial examples of such functions that achieve a given real bond price—simply select a function that takes a sufficiently large value at a sufficiently high realization of inflation. However, it may be less obvious whether plausible economic assumptions can lead to SDFs that look like this. To take one example that anticipates the models to follow, suppose that the SDF depends on the realized price of the long-term bond next period (as would be the case if investors have preferences over wealth). In particular, let the SDF be given by

$$M_{t+1} = P_t^{(1)} \frac{P_{t+1}^{(1)^x}}{E_t\left[P_{t+1}^{(1)^x}\right]} \tag{18}$$

where x is a parameter that can take any real value. This function is strictly positive and has an expected value of $P_t^{(1)}$, so it satisfies the conditions for an SDF. From (14), the price of two-period real bond is

$$P_t^{(2)} = P_t^{(1)} \frac{E_t\left[P_{t+1}^{(1)^{x+1}}\right]}{E_t\left[P_{t+1}^{(1)^x}\right]} \tag{19}$$

It is easy to show that this function is continuous and strictly increasing in the parameter x and that it increases without bound as long as there is positive probability on arbitrarily high values of $P_{t+1}^{(1)}$. Consequently, any specified value of $P_t^{(2)}$, conditional on the current short-term real rate, is attainable by choosing a sufficiently high value of x. The models in the remainder of the paper generate SDFs along these lines, resulting from explicit optimizing behavior on the part of investors, with x interpretable as equilibrium bond holdings. It is through this parameter that quantitative easing will affect real yields, even when nominal yields are stuck at zero.

[7]Note that in this case, where long-term nominal yields are at their lower bound, the real term premium depends negatively on the comovement between the SDF and inflation, a point to which we shall return below.

3 A two-period model

To fix ideas, this section presents a stripped-down model of bond supply and demand in which only one- and two-period bonds exist. The key innovations relative to other models of this type are stochastic inflation and the availability of inflation-indexed bonds. The quantitative model of Section 4 can be thought of as extending the analysis here to a continuum of periods and with more detailed assumptions about the structure of the economy.

3.1 Model setup

Investors enter period 0 of the model with wealth W_0. They have access to four types of zero-coupon assets: 1-period ("short-term") bonds that pay a nominal value of 1 dollar with certainty; 2-period ("long-term") bonds that pay a nominal value of 1 dollar at maturity; 1-period bonds that pay a *real* value of one dollar at the beginning of next period (i.e., one nominal dollar plus the intervening rate of inflation); and 2-period bonds that pay a real value of one dollar at maturity. The 1-period nominal bond (which can also be thought of as interest-bearing bank reserves) is supplied elastically by the government. The two long-term bonds are supplied in inelastic quantities. Let $z^{\$}$ be the quantity that investors demand of long-term nominal bonds (in time-0 dollars) and z be the quantity that investors demand of long-term real bonds. These quantities can take any real values, with negative values corresponding to short positions. (We need no assumption about the supply of real short-term bonds, because the quantity of such bonds demanded will turn out to be indeterminate and have no effect on the results below.)

Anticipating the continuous-time model of Section 4, let i_t be the instantaneous interest rate on the one-period nominal bond in period t and π_t be the instantaneous rate of inflation. The price of the one-period nominal bond in period 0 is thus $P_0^{\$(1)} = e^{-i_0}$, and the gross rate of inflation between periods 0 and 1 is $\Pi = e^{\pi_0}$. The short rate and inflation in period 1 are unknown as of period 0 and have variances σ_i^2 and σ_π^2 and covariance $\sigma_{i,\pi}$. I assume that i_1 is bounded below by b and that π_1 is unbounded from above, but I make no further assumptions about the joint distribution of i_1 and π_1. For the moment, I continue to assume that π_0 is observable at time 0. This assumption is not essential but simplifies the analysis. In particular, since the period-0 real rate is then known with certainty, no-arbitrage implies that short-term real and nominal bonds have the same real return. The Fisher equation $r_0 = i_0 - \pi_0$, where r_0 is the instantaneous real short rate, thus holds.

As above, let $P_t^{(\tau)}$ and $P_t^{\$(\tau)}$ denote the period-t nominal prices of τ-maturity real and nominal bonds, respectively, where $\tau \in \{1, 2\}$, and let $R_1^{(\tau)}$ and $R_1^{\$(\tau)}$ denote the corresponding real gross returns between periods 0 and 1. The following relationships hold by definition:

$$R_1^{\$(2)} = \frac{P_1^{\$(1)}}{\Pi P_0^{\$(2)}} \tag{20}$$

$$R_1^{(2)} = \frac{P_1^{(1)}}{P_0^{(2)}} \tag{21}$$

and, by the Fisher equation,

$$R_1^{(1)} = \frac{R^{\$(1)}}{\Pi} = e^{r_0} \tag{22}$$

Investors' period-1 wealth, in terms of period-0 dollars, is determined by the real return on the portfolio they choose to hold:

$$W_1 = R_1^{\$(2)} z^{\$} + R_1^{(2)} z + R_1^{(1)} (W_0 - z^{\$} - z) \tag{23}$$

Investors are assumed to have mean-variance preferences over real wealth. Thus, in period 0 they solve

$$\max_{(z^{\$}, z)} \mathrm{E}_0[W_1] - \frac{a}{2} \mathrm{var}_0[W_1] \tag{24}$$

subject to (23), where a is a risk-aversion parameter and expectations are taken with respect to time-0 information.

Because they only care about the first and second moments of returns, these investors alone will not enforce the no-arbitrage condition. In particular, they will require two-period bond prices above the bound B^2 if their short position in two-period bonds is sufficiently large and the variance of $R^{\$(2)}$ is sufficiently high. To ensure that no-arbitrage holds, I introduce a second class of participants, "arbitrageurs," who have zero initial wealth and require infinite compensation for downside risk. Thus, arbitrageurs enter the market only when there are risk-free excess returns to be had. In particular, as discussed in the previous section, when $P > B^2$, arbitrageurs can make risk-free profits by shorting two-period bonds and using the proceeds to purchase the one-period risk-free asset. Let $z_{arb}^{\$}$ denote the quantity of nominal bonds demanded by arbitrageurs. Because two-period real bonds always face downside risk (since there is no bound on inflation), arbitrageurs will never hold them.

The model is closed by assuming that the government supplies nominal and real two-period bonds in the quantities $x^\$$ and x. All of these bonds must be willingly held by investors and/or arbitrageurs, so the equilibrium conditions are

$$z^\$ + z^\$_{arb} = x^\$ \tag{25}$$

$$z = x \tag{26}$$

Following other papers in this literature (e.g., Greenwood and Vayanos, 2014) it is implicitly assumed that the government has access to a revenue-generating technology that is sufficient to pay the interest on its bonds in all states of the world without taxing investors. In other words, an unspecified market segmentation exists that allows bonds to be in non-zero net supply from the investors' perspective. Alternatively, $x^\$$ and x can be interpreted as the *net* bond positions held by investors, with positions of equal magnitude but opposite sign held by a separate class of agents, not specified here, who do not maximize the risk/return tradeoff or engage in arbitrage (e.g., the preferred-habitat agents of Vayanos and Vila, 2021.) Negative values of $x^\$$ reflect net lending to the investors in the model by the government or by the unmodeled agents, including, potentially, the investors' future tax liabilities.

3.2 Equilibrium bond returns

Taking the investors' first-order conditions and equating bond supply and demand, the equilibrium expected excess return on nominal bonds must satisfy

$$\mathrm{E}[R_1^{\$(2)}] - R_1^{\$(1)} = a\left((x^\$ - z^\$_{arb})\mathrm{var}[R_1^{\$(2)}] + x\mathrm{cov}[R_1^{\$(2)}, R_1^{(2)}]\right) \tag{27}$$

The required risk premium on a nominal bond depends on the quantities of both real and nominal bonds outstanding. The premium can be negative or positive, depending on whether investors' net exposures are long or short. Similarly, the equilibrium expected excess real return on a real bond is

$$\mathrm{E}[R_1^{(2)}] - R_1^{(1)} = a\left((x^\$ - z^\$_{arb})\mathrm{cov}[R_1^{\$(2)}, R_1^{(2)}] + x\mathrm{var}[R_1^{(2)}]\right) \tag{28}$$

As long as the ELB does not bind for long-term bonds, $z^\$_{arb} = 0$ and the SDF is linear in investors' wealth. Consequently, the no-arbitrage condition (9) becomes

$$x^{\$} - z^{\$}_{arb} \geq \frac{P_0^{\$(1)}(\mathrm{E}_0[P_1^{\$(1)}] - B)}{a\mathrm{var}[R^{\$(2)}]} + x\left(\frac{\mathrm{cov}[R_1^{\$(2)}, R_1^{(2)}]}{\mathrm{var}[R_1^{\$(2)}]} - 1\right) \tag{29}$$

3.3 Real and nominal yields

The time-0 nominal and real yields on two-period bonds, $y_0^{\$(2)}$ and $y_0^{(2)}$, inherit the properties of expected returns discussed above. While yields do not have a closed-form solution in discrete-time, the model is nearly linear so a first-order approximation is quite accurate for realistic parameter values.[8] In particular, when gross returns are close to 1, equation (27) can be written in terms of logs as

$$\mathrm{E}[\log R_1^{\$(2)}] \approx \log R_1^{\$(1)} + a\left((x^{\$} - z^{\$}_{arb})\mathrm{var}[\log R_1^{\$(2)}] + x\mathrm{cov}\left[\log R_1^{\$(2)}, \log R_1^{(2)}\right]\right) \tag{30}$$

I consider the case where the no-arbitrage condition (29) is slack. The case where it binds is less interesting, because nominal yields in that case are zero by construction. Nonetheless, this case will be considered when we discuss the results of the model below. With (29) not binding, making use of (20) - (22) and the definition of bond yields in terms of prices gives

$$y_0^{\$(2)} \approx \frac{\log[e^{i_0}E[e^{i_1}]]}{2} + \frac{a}{2}\left[x^{\$}\sigma_i^2 + x(\sigma_i^2 - \sigma_{i,\pi})\right] \tag{31}$$

This equation holds exactly in the continuous-time limit.

The first term in (31) is the expectations component of the nominal yield, the geometric average of current and future expected nominal short rates. The second term therefore constitutes the nominal term premium, which reflects the compensation that the investor receives for bearing nominal interest-rate risk over the life of the bond. As was the case for bond returns, this premium depends on investors' exposures to both nominal and real bonds. The term premium is zero (i.e., the expectations hypothesis holds) if either risk aversion or nominal interest-rate risk is zero.

A similar calculation shows that the two-period real yield is given by

$$y_0^{(2)} \approx \frac{\log[e^{r_0}E[e^{r_1}]]}{2} + \frac{a}{2}\left[x^{\$}(\sigma_i^2 - \sigma_{i,\pi}) + x(\sigma_i^2 - 2\sigma_{i,\pi} + \sigma_\pi^2)\right] \tag{32}$$

[8] For example, parameters that generate expected net returns on the order of 10% produce second-order approximation terms on the order of one basis point.

The first term is the expectation of real short-term rates over the life of the bond. The second term is the "real term premium," which depends on risk aversion and *real* interest-rate risk. Real interest-rate risk, in turn, depends on the variance and covariance of nominal rates and inflation.

Finally, "inflation compensation" is defined as the difference between real and nominal yields,

$$y_0^{\$(2)} - y_0^{(2)} \approx \frac{\log[e^{\pi_0}(E[e^{\pi_1}])]}{2} + \frac{a}{2}\left[x^{\$}\sigma_{i,\pi} + x(\sigma_{i,\pi} - \sigma_\pi^2)\right] \tag{33}$$

Inflation compensation consists of the expected average rate of inflation over the life of the bond (the first term) and an inflation risk premium (the second term). Depending on the covariance between short rates and inflation, the inflation risk premium could be positive or negative, regardless of whether investors hold long or short positions in each bond type.

Note from equations (31) through (33) that, as uncertainty about the short rate σ_i goes to zero, all elements of the term premia also go to zero, except for the terms $x\sigma_\pi^2$ in equation (32) and $-x\sigma_\pi^2$ in equation (33) . The key feature of the ELB that matters for QE is that it has this effect. In particular, for any value of the expectation of next period's realization of the short rate, the variance of the short rate is bounded by

$$0 \leq \sigma_i^2 \leq \mathrm{E}_0[i_1]^2 - b^2 \tag{34}$$

This implies that the conditional covariance between the short rate and inflation is bounded by

$$|\sigma_{i,\pi}| \leq \sigma_\pi \sqrt{\mathrm{E}_0[i_1]^2 - b^2} \tag{35}$$

These bounds hold regardless of the specific stochastic process determining i_1 and π_1, allowing us to consider the range of potential effects of bond supply on yields near the ELB.

3.4 Quantitative easing and the nominal lower bound

Quantitative easing in this model can be studied as comparative statics across different levels of real and nominal bonds. In particular, a central banker hoping to lower real yields has a choice of two possible policies—reducing $x^{\$}$ by purchasing nominal bonds ("nominal QE") or reducing x by purchasing real bonds ("real QE"). The effects of these policies in different states of the world can be summarized by taking the derivatives of yields with respect to $x^{\$}$ and x in equations (31) - (33).

I emphasize four results:

Result 1 *As long as $\sigma_{i,\pi}/\sigma_\pi^2 > 1$, a given amount of nominal QE lowers the long-term real yield by more than the same amount of real QE does.*

This result follows immediately by comparing the two pieces of the term premium in equation (32). Note that the condition $\sigma_{i,\pi}/\sigma_\pi^2 > 1$ will generally be satisfied if the central bank responds aggressively to inflation. In particular, if the central bank strictly follows the Taylor principle—moving the short rate more than one-for-one with inflation and never deviating from that rule—the condition will be met. We thus have the (perhaps counterintuitive) result that, under normal circumstances, buying nominal bonds has a bigger effect on real yields than buying real bonds.

Result 2 *Nominal QE has no effect on either nominal or real yields when the long-term nominal yield is at the lower bound.*

There are two ways that $y_0^{\$(2)}$ can reach the lower bound. First, the expectation of the period-1 short rate could go to b. In this case, Result 2 must hold because, by (34) and (35), the variance of the short rate and its covariance with inflation both go to zero as the expectation of next period's short rate shrinks to the ELB. And, from (31) and (32), as these variance and covariance terms go to zero, all elements of term premia also go to zero, except for the term $x\sigma_\pi^2$ in equation (32). Thus, when tomorrow's 1-period yield is known with certainty, the quantity of nominal bonds has no effect on yields. The second possibility is that $E_0[i_1] > b$ but the nominal term premium $\frac{a}{2}\left[x^{\$}\sigma_i^2 + x(\sigma_i^2 - \sigma_{i,\pi})\right]$ takes a negative value such that the nominal yield goes to b. But in this case the no-arbitrage condition (29) implies that further reductions in the nominal term premium are impossible. Arbitrageurs would step in and exactly offset the effects of any nominal QE purchases at this point by shorting an equal quantity of nominal bonds to investors. Consequently, in equilibrium, the net nominal exposures of investors $x^{\$}$ would be unchanged, and there would be no effect of the purchases on real or nominal yields.

Result 3 *The effect of real QE on the long-term real yield is strictly negative, even when nominal yields are at their lower bound.*

Result 3 simply reverses the logic of Result 2. If the expected nominal short-rate is b, so that $\sigma_i = \sigma_{i,\pi} = 0$, equation (32) shows that the effect of real bonds on real yields is $a\sigma_\pi^2$, a strictly positive quantity. If, on the other hand the expected nominal short rate is greater than b, the multiplier on x in equation (32) is still strictly positive, and the arbitrage intervention that applies to nominal bonds does not occur here because arbitrageurs will not take positions in real bonds.

Result 4 *There is some value $> b$ of the nominal yield below which real QE is more effective at reducing the real yield than nominal QE is.*

This is an immediate implication of Results 3 and 4. Note that it follows that, below the same threshold, the effect of real bond purchases on inflation compensation is necessarily positive. These are two sides of the same coin. When nominal short rates are near the ELB, real rates and inflation move in opposite directions and consequently command opposite-signed risk premia. Indeed, one way of understanding why real bond purchases must reduce real term premia at the ELB is to note that these purchases remove a good inflation hedge from investors' portfolios, increasing the compensation they require for inflation risk. Since nominal yields are stuck at zero, real yields must fall to let the inflation-risk premium rise. Note also that the negative correlation between real rates and inflation in this situation reverses the sign that applies under the Taylor principle. This is what causes the relative efficacy of nominal and real QE to switch. The Taylor principle *must* cease to hold as the ELB approaches because *at* the ELB short rates do not move in response to inflation at all. Indeed, one can show that the covariance of nominal rates and inflation must be negative when $\mathrm{E}_0[i_1] < b + \sigma_\pi$.

Figure 2 illustrates some of these results graphically. It depicts the the ranges of possible derivatives of $y^{\$(2)}$ and $y^{(2)}$ with respect to $x^{\$}$ and x for an arbitrary value of σ_π.[9] The horizontal axis shows values of $\mathrm{E}_0[i_1]$. (Because these effects operate only through the term premium, the *current* short rate i_0 is irrelevant.) The effects of nominal bond quantities on nominal yields are strictly positive, and they are larger when short rates are more volatile—the standard result of Vayanos and Vila (2020) (panel A). As in King (2019), these effects generally decline to zero as the ELB is closer to binding because the volatility necessarily decreases. The effect of nominal

[9] Of course, it may be the case that the variance of inflation also changes as interest rates approach the ELB. However, because the results hold qualitatively for *any* positive inflation variance, they will also hold qualitatively regardless of how this variance fluctuates.

bond quantities on real yields (panel B) can be positive or negative and also generally declines as we approach the ELB. Nominal bond supply may have a greater or smaller effect on nominal yields than on real yields, depending on the correlation between i and π. Consequently, inflation compensation may be increasing or decreasing in nominal bond supply (panel C). The important observation, however, is that the effects of nominal bonds on nominal yields, real yields, and inflation compensation all go to zero as $\mathrm{E}_0[i_1] \to b$.

The second column of charts shows the effects of real bond supply. As shown above, the effect of real bonds on nominal yields in this simple model is the same as the effect of nominal bonds on real yields. (Panel D mirrors panel B.). Thus effects of real bonds on nominal yields also necessarily go to zero for small values of $\mathrm{E}_0[i_1]$. Panel E shows that the effects of real bonds on real yields are strictly positive everywhere. Unlike nominal bonds, real bonds continue to have an effect on real yields even when the nominal short rate is at the ELB with certainty. In that case, the effect on the real term premium of a marginal change in x is equal to $a\sigma_\pi^2$. Finally, although an increase in real bonds can have positive or negative effects on the inflation risk premium in general, it necessarily decreases the inflation risk premium when $E[y^{\$(1)}]$ is low enough. At the ELB the effect of real bonds on the inflation risk premium is $-a\sigma_\pi^2$, the opposite of its effect on the real term premium.

The following section adds several bells and whistles to this simple two-period model, but the qualitative results just described continue to hold, and for the same fundamental reasons.

4 Quantitative model

I now extend the two-period model above to a somewhat more realistic setup, allowing for comparisons to the data and quantitative evaluation of QE-like programs. In particular, I now allow investors to have access to a spectrum of real and nominal bond maturities and I provide an explicit process for the joint dynamics of short-term rates, inflation, and real activity. In the model, inflation and the output gap are assumed to depend, in part, on the level of the long-term real yield, allowing for feedback between the yield curve and the real economy and thus for the transmission of QE-type policies. While the macro block of the model is specified as a reduced-form process, it can be motivated by structural models in which economic activity depends on long-term yields, such as Ray (2019).

4.1 Model description

Investors have access to a continuum of zero-coupon nominal bonds and a continuum of zero-coupon inflation-indexed bonds, each with maturities 0 to T. At maturity, nominal bonds pay a nominal value of \$1, while real bonds pay a nominal value of \$1 times the change in the price level that has accrued over their lifetimes. At each point in time t, investors choose to hold a market-value quantity $z_t^{\$}(\tau)$ of nominal bonds and $z_t(\tau)$ of real bonds of each maturity τ. Let $P_t^{(\tau)}$ represent the time-t price of a bond with remaining maturity τ. In addition, investors have access to a risk-free security that pays the instantaneous rate r_t. Investors' time-t (real) wealth W_t is the sum of the market-value of the bond portfolio and the risk-free asset, and it thus evolves according to

$$\begin{aligned} dW_t = & \int_0^T \left[z_t^{\$}(\tau) \left(\frac{dP_t^{\$(\tau)}}{P_t^{\$(\tau)}} - \pi_t \right) + z_t(\tau) \frac{dP_t^{(\tau)}}{P_t^{(\tau)}} \right] d\tau \\ & + \left(W_t - \int_0^T \left[z_t^{\$}(\tau) + z_t(\tau) \right] d\tau \right) r_t dt \end{aligned} \tag{36}$$

Taking W_t as given, investors choose quantities $z_t^{\$}(\tau)$ and $z_t(\tau)$ to solve the problem

$$\max_{\{z_t^{\$}(\tau), z_t(\tau)\}\forall\tau} \mathrm{E}_t[dW_t] - \frac{a}{2}\mathrm{var}_t[dW_t] \tag{37}$$

subject to (36).

The government exogenously supplies quantities of nominal and real bonds, $x_t^{\$}(\tau)$ and $x_t(\tau)$, at each maturity. A solution to the model is a set of state-contingent bond prices that clear the market. Specifically, market clearing requires

$$z_t^{\$}(\tau) = x_t^{\$}(\tau) \tag{38}$$

$$z_t(\tau) = x_t(\tau) \tag{39}$$

at each maturity τ and at each point in time t. Prices adjust to ensure these equations hold in all states of the world, given investors' optimization. Because I will assume that there are no discrete jumps in state variables, the no-arbitrage condition (9) is always satisfied, and there is no need to specify a separate demand function for arbitrageurs.

I assume that the nominal short rate i_t is determined by

$$i_t = \max[\widehat{i}_t, b] \tag{40}$$

where the shadow rate $\widehat{i}_t$ follows the process

$$\begin{aligned} d\widehat{i}_t &= \kappa_i(\mu^i_t - \widehat{i}_t)dt + \sigma_i dZ^i_t && (41) \\ \mu^i_t &= r^* + \pi^* + \phi_{i,\pi}(\pi_t - \pi^*) + \phi_{i,g}(g_t - g^*) && (42) \end{aligned}$$

with g_t being the time-t output gap and Z^i_t a Brownian motion. The parameters r^*, π^*, and g^* represent the long-run means of the real short rate, inflation, and the output gap. This process makes the short rate obey an inertial Taylor rule with a lower bound, subject to random deviations.[10]

I assume that the reduced-form processes determining the instantaneous changes in the output gap and inflation are linear in the levels of those variables and in the current level of the 10-year real yield:

$$\begin{aligned} dg_t &= \kappa_g(\mu^g_t - g_t)dt + \sigma_g dZ^g_t && (43) \\ \mu^g_t &= g^* + \phi_{g,\pi}(\pi_t - \pi^*) + \phi_{g,y}(y^{(40)}_t - y^{(40)*}) && (44) \end{aligned}$$

and

$$\begin{aligned} d\pi_t &= \kappa_\pi(\mu^\pi_t - \pi_t)dt + \beta dZ^g_t + \sigma_\pi dZ^\pi_t && (45) \\ \mu^\pi_t &= \pi^* + \phi_{\pi,g}(g_t - g^*) + \phi_{\pi,y}(y^{(40)}_t - y^{(40)*}) && (46) \end{aligned}$$

where $y^{(40)*}$ is the (endogenously determined) steady-state value of the 40-quarter real yield and Z^g_t and Z^π_t are additional, independent Brownian motions. These equations will allow for passthrough to the macroeconomy both from shocks to the shadow rate that change the expectations component of yields and from QE-type policies that change term premia.

Finally, in order to reduce the dimension of bond supply across maturities, I assume that the maturity distributions of nominal and real bond supply are flat over maturities of zero to 15 years with density given by the parameters $x^\$$ and x, respectively (and

[10] Although the ELB is imposed *a priori* here, it is trivial to extend the model to endogenize it by allowing investors to hold an elastic supply of cash (paying zero nominal return) in addition to the risk-free asset.

equal to zero at maturities longer than 15 years). Because these are zero-coupon bonds, whose duration is equal to their maturities, the 15-year bond has a duration close to the maximum duration typically available for coupon Treasuries in the U.S.. The flat maturity structure has the advantage that the nominal and real dollar-duration held by investors, which is what matters for bond pricing, are proportional $x^{\$}$ and x.[11] Since I do not want to assume that the relevant investors in the real world hold *all* Treasury bonds or *only* Treasury bonds, I do not attempt to match $x^{\$}$ and x to actual bond-quantity data. Rather, I will calibrate those parameters to ensure that the model achieves realistic levels of long-term yields.

Because of the nonlinearity, the model does not admit an analytical solution. I solve it numerically, using a method similar to that in King (2019). In brief, I solve the model globally by discretizing the state space and iteratively (a) calculating state-contingent bond prices by solving the model (36) - (39) using a given set of conditional expectations, and (b) numerically calculating conditional expectations given state-contingent prices using the transition densities implied by equations (40) - (46). Cubic interpolation between the discretized nodes is used for situations, such as model simulation, in which state values are required to be continuous.

4.2 Parameter values

I parameterize the model based on quarterly observations on zero-coupon nominal U.S. Treasury yields and TIPS from the Gurkaynak et al. (2007, 2008) datasets, headline CPI inflation, and the CBO output gap over the period Q1:1999 - Q4:2020. The beginning of the sample is determined by the availability of the TIPS data. Parameter values are given in Table 1.

There are several sources of noise in the raw series that make it problematic to estimate the model directly on them. For example, headline CPI inflation (which is the series TIPS are indexed to) displays a large amount of high-frequency variation that has essentially no forecasting power and little relevance for monetary policy. In addition, neither CPI nor the CBO output gap are the precise series that the Federal

[11] In reality, the maturity distribution of Treasury securities is downward sloping, rather than flat, with large outstanding amounts at the short end and little beyond 20 years. Although one could approximate this shape with a suitable function, the precise configuration of maturities typically has very small quantitative effects in these types of models since what matters is the total duration held by investors, not how it is distributed (see King, 2018). It is also unclear what the relevant shape should be, since the marginal investors in reality may not hold bonds in proportion to the total amount outstanding.

Reserve bases monetary policy on, and the T-bill rate is not exactly the same as the monetary-policy instrument. For all of these reasons, I assume that the CPI, output gap, and T-bill rate are measured with iid error, and that only the underlying latent series drive the dynamics of the system.

In addition, to aid in the identification of the parameters of interest, I calibrate a few values to external sources. First, for the parameters of the monetary-policy rule (40) - (42), I use the standard Taylor (1993) values, together with the inertia of 0.76 estimated by Carlstrom and Fuerst (2008). These values are hard to estimate precisely over the last 20 years, in part because of the significant amount of time spent at the ELB. I set the value of the ELB itself to $b = 0$. Second, since unconditional means of persistent series are weakly identified in small samples, I assume that the long-run means of inflation and the output gap are $\pi^* = 2.4\%$, and $g^* = -0.5\%$. The first is the average expected value of CPI inflation, over a ten-year horizon, reported in the Blue Chip Economic Indicators over the period 1999 - 2020; the second is the average value of the CBO output gap since its inception in 1948. Finally, I approximate the level of $r*$ in every quarter by the 6- to 11-year forecast of Tbill rates minus the corresponding forecast of inflation in the Blue Chip Economic Indicators survey.

Conditional on these values, I estimate the remaining parameters of the model—including the key parameters that determine how inflation and output respond to real yields. The joint dynamics of observed CPI, output gap, and the T-bill rate are given by a nonlinear state-space model with three latent factors. I estimate the model using a particle filter with Gibbs sampling and use posterior means of the parameter distributions. Details of the estimation are described in Appendix B. Most notably, the estimates of $\phi_{\pi,y}$ and $\phi_{g,y}$ are both negative, implying that decreases in long-term real yields have stimulative effects on both inflation and output in the short run.

Finally, the values of $x^{\$}$, x, and $r*$ are calibrated separately to match the levels of yields that are relevant as initial values in each of the three scenarios described below.

5 Policy Experiments

To examine the effects of nominal versus real QE programs in different plausible environments, I perform the following experiment. Given a set of initial conditions that approximate the average conditions that prevailed when the first Treasury QE program was launched in the U.S. in late 2008 and 2009, I find the quantity of nominal bond purchases that reduces the ten-year nominal yield by 100 bp. Then, given the same conditions, find the changes in yields that would have resulted by purchasing the same quantity of real bonds instead. I then change the initial conditions to approximate the average conditions that prevailed in 2020, as the Federal Reserve again hit the ELB and implemented QE in response to the COVID-19 crisis, and in a third scenario in which $r*$ has fallen further. I consider equal-sized QE programs in both real and nominal bonds in these three cases and compare their effects.

5.1 Initial conditions

For ease of reference, I call the three cases I will consider the "high rate," "moderate rate," and "low rate" scenarios. The parameter values and initial state-variable configurations that define each scenario are summarized in Table 2. All three scenarios envision a recession in which the short-term nominal rate is reduced to the ELB. In particular, I set the state variables to the following values in the initial period ($t = 0$) in each scenario: $\widehat{i}_0 = 0$; $\pi_0 = 0.01$; $g_0 = -0.04$. These values are roughly those that were observed around the time that the short rate was cut to the ELB and QE policies were enacted in the U.S. in each of the last two recessions.[12]

The differences across the scenarios are due to the initial levels of long-term yields, which depend on the values of $x^{\$}$, x, and $r*$. In the high-rate scenario, I set $r*$ equal to 1.7%, its value in December 2008 according to the survey-based evidence mentioned above. In the moderate-rate scenario, I set $r*$ equal to 0, the survey-based value in March 2020. In the low-rate scenario, I envision a future in which $r*$ has fallen by the same amount again, to a level of -1.7%. In the high- and moderate-rate scenarios, I calibrate the values of $x^{\$}$ and x such that the nominal and real ten-year yields are near the values that were observed around the previous periods when QE was implemented in the U.S.. In particular, in the high-rate scenario the initial value of the nominal and real ten-year yields, respectively, are set to 4.3% and 1.6%. In the moderate-rate scenario, they are set to 1.4% and -0.6%. The values of $x^{\$}$ and x that are consistent with these yields are shown in the table.[13] Since there are no data to guide the choice

[12]According to the state-space model, the point estimates of latent inflation and the output gap were 1.2% and -4.8% in December 2008 and 0.5% and -3.5% in March 2020. Although one could tune the starting values in the scenarios to those exact values to more carefully dissect those two historical episodes, that is not the central purpose here. Instead, letting these values be the same across all three scenarios helps isolate the effects of changes in bond supply and facilitates comparability of QE policies.

[13]Because of dislocations in the Treasury market—particularly in the TIPS market—around these

of the bond-supply parameters in the low-rate scenario, I set them equal to the same values used in the moderate-rate scenario. This produces initial values for the ten-year nominal and real yields of 0.0% and -1.0%, respectively.

5.2 Shock calibration

To establish a baseline, I first find the reduction in nominal bond supply $x^{\$}$ that causes the nominal ten-year yield to fall by exactly 100 bp at time 0 in the high-rate scenario. (The size of the reduction itself has no interpretable meaning but turns out to be equal to 0.055—about 12% of the time-0 nominal duration exposure.) I then ask what the effects of this same change in bond supply would have been if it had instead been applied to real, rather than nominal, bonds and if it had occurred in the other two scenarios. This allows me to compare the "bang for the buck" of alternative QE policies in different environments.

A complicating issue is that the model allows for feedback between the macro variables and the short-term rate. Without further adjustments, a QE shock that lowers real yields will raise GDP and inflation, triggering an increase in the short rate through the Taylor rule over time. Because investors in the model have rational expectations, they will anticipate this increase as of period 0, which will cause the expectations component of long-term yields to rise immediately, offsetting at least part of the initial downward shock. Both to facilitate the analysis and to add realism, it is helpful to shut down this feedback.[14] Thus, for each scenario, the baseline simulation is one in which a negative shock to the shadow rate i_0 occurs at the same time that the QE shock occurs. The size of the shadow-rate shock is chosen such that the expectations component of the ten-year nominal yield is unchanged in the initial period. As shown in the last two rows of the table, because expectations of future short rates are not very sensitive to the current value of the shadow rate at the ELB, it sometimes requires very large shadow-rate shocks to neutralize the feedback. Nonetheless, the effects on

times, it is not straightforward to get a clean read on exactly what the equilibrium level of yields actually was. The values that I use are approximations based on informally smoothing through the higher-frequency fluctuations in 2008-09 and 2020. Again, however, the precise values are not important because the purpose here is not to exactly reconstruct the past episodes but rather to consider QE policies in different plausible states of the world.

[14]This "adds realism" because it seems unlikely that a central bank would want to increase short rates in response to the stimulus provided by its own QE program. Indeed, if anything, evidence indicates that policy expectations shifted *down* in response to QE announcements, through the so-called "signaling channel," as discussed in Section 5.4.

the realized values of the other variables will turn out to be relatively modest. For comparison, I also show the case in which these neutralizing shocks do not occur and policy expectations are allowed to respond endogenously to the QE shocks.

5.3 Results

5.3.1 Yields

Figure 3 shows the responses of the real and nominal yield curves in the period when the shock occurs in each of the three scenarios. The initial configuration of the yield curves is shown as the solid black lines, while the post-shock curves are shown as dashed lines. The blue dashed line is the response to QE purchases of nominal bonds, while the red dashed line is the response to equal-sized purchases of real bonds. In the left-hand panels, the feedback from macro variables to the short rate is neutralized by introducing the additional shadow-rate shock, as described above. In the right-hand panels, the short rate mechanically follows the usual policy rule. To facilitate comparison, Table 3 reports the initial reactions of 10-year yields and inflation compensation in each case.

Beginning with the left-hand panels, the ten-year nominal yield in the high-rate scenario (panel A) falls by exactly 100 bp in response to the nominal-QE shock, by construction. The ten-year real yield falls by 86 bp in response to this shock. Inflation compensation declines by 14 bp (in spite of the fact that expected inflation rises because of the stimulus), reflecting the transfer of nominal risk from investors to the government. Empirical event studies on the Fed's first round of QE programs generally indicate that such programs caused a *larger* response of real yields than nominal yields (Krishnamurthy and Vissing-Jorgensen, 2012; Abrahams et al., 2016). The discrepancy between that evidence and the model results can be reconciled by noting that, to focus the analysis, the experiment here omits any "signaling channel" of QE purchases. Section 5.4 extends the analysis to include the signaling channel, and doing so produces a larger real-yield response, consistent with the evidence, but does not change the other results of the model significantly.

In response to a real QE shock of the same magnitude as the nominal QE shock just considered, the real yield falls by just 58 bp, rather than 86. As discussed above, the larger response of the real yield nominal QE shocks is consistent with the large, positive correlation between nominal rates and inflation induced by the Taylor principle in the model, since, in the high-rate scenario, ELB outcomes that violate that principle are rare.[15]. When we move to the moderate-rate scenario (panel B), the same quantity of nominal or real QE has smaller effects on both real and nominal yields (and inflation compensation) than in the corresponding cases above, declining by 10% to 40%. However, it remains the case that purchases of nominal bonds have a greater effect on real yields than purchases of real bonds.

In the low-rate scenario, things are quite different (panel C). Because the nominal yield curve in this case begins the simulation flat at zero across all maturities, QE has no effect on nominal yields, regardless of whether it is implemented through real or nominal bonds. Moreover, nominal QE no longer has any effect on real yields. However, the effect illustrated in the two-period model kicks in: real bond purchases continue to cause real yields to fall. Indeed, the effects of real QE on the ten-year real yield are almost as large in the low-rate scenario as in the moderate-rate scenario (-30 bp versus -38 bp). As in the low-rate outcomes studied in the two-period model, inflation compensation rises in response to a real QE shock here, since real bonds, which hedge inflation in this environment, are removed from investors' portfolios.

The simulations where the short-rate feedback is not neutralized lead to notably smaller responses of real yields in the high- and moderate-rate scenarios. For example, the decline in the real yield in response to the nominal QE shock in the high-rate scenario is 44 bp, rather than 86, when the feedback is allowed. Despite these quantitative differences, however, the qualitative patterns in yields remain the same in the non-neutralized scenarios. It continues to be the case that nominal QE has a bigger effect on real yields than real QE above the ELB, but these effects decline and eventually disappear as rates get low. It also remains true that real QE has a significant effect on real yields even in the low-rate environment. Indeed, in the low-rate scenario, there is nothing to neutralize, because nominal short-rate expectations remain at the ELB even after the QE shocks.

5.3.2 Macro variables

Figures 4 - 6 show impulse-response functions for yields and the macro variables over the subsequent five years in each scenario.[16] In each panel, the gold line shows the

[15]Recall that in the two-period model the response of real yields to nominal QE was equal to the response of nominal yields to real QE. Because of the additional complexities of the quantitative model, that correspondence is not exact here, but the values are nonetheless similar to each other in all of the scenarios.

[16]IRFs are calculated by simulating the model from the initial conditions given a QE shock and subtracting the corresponding values from a simulation with the same initial conditions but in which

baseline model in which feedback through the Taylor rule is neutralized using shadow-rate shocks, while the blue line shows the simulation in which the feedback is allowed. To facilitate comparison, Table 3 summarizes the effects by reporting the values of the macroeconomic variables after 10 and 20 quarters.

The results are what one would expect given the shocks' relative impacts on real yields. In the high-rate scenario, nominal bond purchases have relatively large macroeconomic effects, though they take some time to fully materialize. In particular, the nominal QE shock increases inflation by about 30 bp and the level of output by about 60 bp in the medium-term. (These values are only about 20 and 40 bp, respectively, when the feedback through the short rate is not neutralized.) Purchases of real bonds in this scenario have smaller, but still notable, effects. Again, in the moderate-rate scenario, the macroeconomic results are qualitatively similar but a bit weaker.[17]

In the low-rate scenario, nominal QE purchases have no macroeconomic effects, since they do not reduce real yields. Meanwhile, real bond purchases, which manage to reduce the real yield by 30 bp even in this environment, continue to induce significant responses in output and inflation. Indeed, the macro responses are comparable to the magnitudes generated by the same shock in the high- and moderate-rate environments, in spite of the real yield response being somewhat lower. For example, after 5 years, inflation rises by 17 bp, and the level of output rises by 35 bp.

5.4 Allowing for the signaling channel

To keep the analysis focused on the effects of QE through term premia, the exercises above all assumed that asset purchases have no effect on agents' expectations for the short-term interest rate (except endogenously through the Taylor rule in the simulations where feedback is allowed). But a number of studies have suggested that signaling a central bank's commitment to keeping the future short-term rate low may be a major channel through which QE operates (e.g., Bauer and Rudebusch, 2014; Battarai et al., 2015). I now re-run the experiments allowing for this channel.

Signaling in the model is accomplished by assuming that a negative shadow-rate

no QE shock occurs.

[17] Note that, in the baseline simulations for both the high- and moderate-rate scenarios, the shadow rate falls substantially in order to neutralize the endogenous effects of QE on the policy rate path. Because the simulations begin with the shadow rate at the ELB, the additional decline implies that the nominal short rate will remain equal to the ELB for several quarters, which, since it is taken relative to a simulation in which there is no QE shock, implies a monotonic decline in the IRF during this period.

shock occurs in the same period in which the central bank purchases bonds. Empirical evidence on the magnitude of signaling effects versus duration effects is mixed. For illustration, I assume that signaling accounts for half of the effect of QE on nominal yields in the high-rate regime. That is, I consider a reduction in outstanding nominal bonds that lowers the 10-year nominal yield by 50 bp in the high-rate scenario, and a simultaneous shock to $\widehat{i}_0$ that lowers that yield by an additional 50 bp.[18] Thus, the initial movement on the nominal yield is the same as in the experiments above, but that movement is now equally the product of lower short-rate expectations and reduced duration, rather than reduced duration alone.

The first column of Table 4 shows the results (cf. the first column of Table 3A). The first thing to note is that, unlike the baseline analysis above, QE now has a larger effect on long-term real yields than on long-term nominal yields. This is because the signaling channel results in an increase in expected inflation. Consequently, the expected path of r_t falls by more that the expected path of i_t. As noted earlier, empirical event studies, which are largely based on data from a regime similar to the high-rate scenario, tend to suggest that real yields did indeed fall by more than nominal yields in response to QE shocks. So allowing for a signaling channel can bring the model into closer alignment with the data in this sense.

In spite of the fact that real yields initially fall by more here than in Table 3, however, there is little difference in macroeconomic outcomes. The reason is that, although it is bigger in period 0, the reduction in real yields that occurs through the signaling channel is shorter-lived than the reduction that occurs through the duration channel. The bigger initial effect and lower persistence roughly offset in terms of their medium-term effects on output and inflation.

The remaining columns of the table translate the QE purchases to real bonds and examine the effects across the other two interest-rate environments, just as in Table 3. When comparing nominal versus real QE, I continue to use an equal reduction in bond quantities for both, as above. I also assume that the signaling effects of real QE are the same as the signaling effects of nominal QE and are invariant to the initial level of interest rates—that is, the assumed shadow rate shock is the same across all six columns of the table.

The signaling channel becomes less effective in lower-interest-rate environments.

[18]The size of the shocks that accomplish this are -0.033 for $x^{\$}$ and -170% for $\widehat{i}_0$. For interpretation, the shadow-rate shock pushes back the modal timing of policy rate "liftoff" by 15 quarters.

This is intuitive, since signaling about policy rates over a given period can accomplish little if market participants already expect those rates to be close to the ELB during that entire period. In the extreme, where the ten-year yield is already at the ELB, signaling has no effect. Since, for the reasons discussed above, nominal bond purchases also have no effect through the duration channel in that scenario, QE is impotent regardless of whether the signaling channel is allowed or not.

Otherwise, there is not much qualitative or quantitative difference between these scenarios that include the signaling channel and those shown above that exclude it. The effects of real bond purchases are smaller, in terms of basis points, in the low-rate environment, but this is because the assumed reduction in bond purchases itself is smaller. Most importantly, real QE continues to have non-trivial effects on the macroeconomy, even when the potential of nominal QE has been exhausted.

From these observations, I conclude that purchases of real bonds could continue to provide some amount of stimulus for the economy, even when nominal yields are driven to their lower bound across all maturities. Of course, it may require very large purchases of such bonds to have the desired effect, and in most jurisdictions the size of the market, if it exists at all, is too small to allow this. However, economically equivalent types of policies may be available. I briefly turn to this issue next.

6 Practical considerations

As noted earlier, inflation-linked bond markets are a small fraction of overall government debt markets in most advanced economies. Thus, implementing purchase programs of such bonds at a large scale is generally infeasible. However, an asset-purchase program is not the only way to create the outcomes that the model prescribes. From the perspective of the model, what matters is that investors receive a short exposure to inflation risk, or, equivalently at the ELB, that their net holdings of real duration decrease. This outcome can be achieved if, instead of selling real bonds, the investors borrow at an inflation-indexed rate. In terms of the net exposures x, holding negative quantities of bonds is the same as taking out a loan. Consequently, in principle, a central bank could put downward pressure on real yields by lending in real terms to financial institutions.

While a full discussion of such a lending program is beyond the scope of this paper, we can make a few preliminary observations. First, to be effective in removing real duration, inflation-indexed lending would have to have a maturity of several years.[19] Lending at such long horizons would not be unprecedented. A number of major central banks, including the ECB and the Bank of England, have offered credit to banks at nominal rates at long maturities in the recent past.[20] Section 10 of the Federal Reserve Act prohibits the Fed from lending at terms of more than four months under normal circumstances. There is an exception for loans against residential mortgage collateral, but in those cases the rate is specified to be the lowest that applies to any Reserve Bank credit. Thus, most likely, an invocation of Section13(3) of the Act, requiring the consent of the Treasury, would be required to implement such a program under the current legal structure. But these emergency powers are not such exceptional events—they have successfully been used in each of the last two recessions.

Second, as with nominal lending, a central bank would have options concerning how to structure an inflation-indexed lending program, depending on the relative value it places on precise control over the level of yields, the size of its balance sheet, and other factors. For example, such loans could be made in fixed quantities, through an auction mechanism, or at fixed real rates, as a standing facility. In either case, from an accounting perspective, the operations would be essentially equivalent to a real bond-buying program from the central bank's perspective just as they would be from investors' perspective. Instead of holding a bond on the asset-side of its balance sheet, the central bank would hold a loan. Reserves in the banking system would increase by an amount equal to the increase in assets, just as with QE operations.

The economics behind such a hypothetical lending facility are straightforward. Consider a traditional bank that makes loans and takes deposits in nominal terms. To a first approximation, such a bank is hedged against inflation shocks; both its real loan revenue and its real deposit costs fall when inflation rises. But if this bank replaces its deposits with inflation-indexed term credit, it is no longer hedged. A positive shock to inflation will reduce its real income but have no effect on its real interest expense. All else equal, in order to accept this extra inflation risk, the bank will require that the real rate on its liabilities must be lower. If the central bank were to auction a large amount of such credit, the stop-out rate could well be deeply negative—precisely the desired result. Of course, it is also likely that financial institutions would try to pass

[19]The loans would also have to prohibit or impose steep penalties on prepayment.

[20]The BoE offered credit to banks at terms up to four years through the Funding for Lending Scheme and the Term Funding Scheme. The ECB offered loans up to three years through its Long-Term Refinancing Operations. The Reserve Bank of New Zealand, the Reserve Bank of Australia, and the Riksbank have all also offered loans up to three years through various lending facilities.

on the inflation risk to their customers by issuing inflation-indexed loans themselves or engaging in derivatives transactions. But this would be the transmission mechanism at work.

Inflation-indexed term lending would have some advantages over negative nominal short-term rates, which is an alternative that some central banks have experimented with after their policy rates reached zero (see, e.g., Campbell et al., 2020). One objection to negative nominal short rates is that they may be costly for banks and therefore spur disintermediation or excessive risk-taking (Brunnermeier and Koby, 2018; Heider et al., 2019). But these outcomes would be less likely with inflation-indexed term lending because banks' funding costs would be reduced in real terms. Another concern with negative nominal short rates is that they could cause problems for entities like money-market mutual funds, which require positive nominal returns on short-term debt to remain operational (Di Maggio and Kacperczyk, 2017; Arteta et al., 2018). But inflation-indexed lending would not cause negative nominal rates, so this concern wouldn't apply. Indeed, if it implemented such a program, a central bank could potentially *raise* its short-term policy rate slightly to provide relief to such institutions. Finally, some have noted that very negative nominal policy rates may not be attainable at all, at least as long as paper currency exists, because at some point agents will just hoard cash rather than accept negative nominal returns (e.g., Rogoff, 2017). But the model makes clear that this possibility does not apply to inflation-indexed loans. Those instruments cannot be arbitraged with cash because they involve a risk, and a risk premium, that cash does not.

A separate question is whether the economy would respond to changes in long-term real yields that were not accompanied by changes in nominal yields or expected inflation. One might question the possible extent of pass-through, since private issuance of inflation-linked debt is generally very small. But with high inflation-risk premia and low real term premia, firms would have strong incentives to issue inflation-indexed bonds, or to engineer their synthetic production through the use of derivatives.[21] Banks too would have incentives to offer inflation-indexed loans, since, as just noted, they would otherwise be left with unhedged inflation exposure on the asset side of their balance sheets. Finally, borrowers should be eager to take inflation-indexed credit,

[21]Issuance of standard, nominal bonds and loans is equivalent to issuance of inflation-indexed products if it is accompanied by an inflation swap between the borrower and the originating financial institution. Inflation swaps markets are larger in Europe than in the United States, but again there would be incentives for participants to grow these markets in the scenarios envisioned here.

since by doing so they would expect to pay real interest rates much lower than they would pay on nominal credit (albeit only by accepting greater liability-side inflation risk).

Apart from issuance of private inflation-indexed debt, real term-premium shocks can be transmitted to the economy in other ways. For example, they should also work directly on asset prices, outside of intermediation channels, thereby stimulating the economy through wealth effects. The prices of inflation-linked bonds, one small component of consumer wealth, are directly affected by a QE program that purchases them. But the reduction in the real term-premium should also pass through to any other asset that acts as a partial inflation hedge (i.e., for which the inflation-risk premium does not rise one-for-one with reductions in the real term premium at the ELB). For example, real returns on some types of equities are typically resilient against inflation, since increases in the nominal prices that firms charge customers are ultimately passed through to higher nominal dividends. Thus, all else equal, decreases in the real term premium should lift stock prices, while increases in the inflation-risk premium should have little effect on them. Similar arguments can be made for a variety of assets that pay off in real values, such as commodities and housing. They may also apply to nominal private debt with credit risk, since borrowers ability to pay such debt is positively correlated with inflation.

7 Conclusion

The evidence in Gilchrist et al. (2015), Gertler and Karadi (2015), and others strongly suggests that changes in long-term real term premia can have important effects on the macroeconomy. This paper has studied the extent to which QE programs can affect these premia when the lower bound is close to binding on the nominal yield curve. In that situation, purchases of nominal bonds have a small effect on both real and nominal yields that disappears altogether when the lower bound is reached. But purchases of real bonds can reduce real yields without limit. They do so by transferring inflation risk to the private sector, driving the inflation-risk premium up and the real term premium down in equal measure. The results of the quantitative model, which captures key features of the data well, suggest that buying real bonds—or other economically equivalent transactions that a central bank could undertake—could be effective in lowering real yields and stimulating economic activity in future episodes where the nominal lower bound binds at long maturities.

From an academic standpoint, this paper contributes to the literature, introduced by Vayanos and Vila (2021), studying the effects of bond supply held by investors on the term structure of interest rates. Most of that literature has ignored both inflation and the effective lower bound, two elements that are central to capturing the behavior of the yield curve and its response to QE-type policies. I incorporate both features, as well as real economic activity, into such a model and introduce a distinction between real and nominal bonds.

8 Appendix A. Solution to quantitative model

This appendix describes the solution for bond prices in the quantitative, continuous-time model of Section 4. Because of the nonlinearities, an analytical solution is not available. However, we can characterize the equilibrium conditions implicitly in terms of the covariances of bond prices. These covariances are endogenous in the model and are found numerically using the method of King (2019) (see text).

The first-order conditions for this problem can be written in terms of the expected excess real returns on real and nominal bonds:

$$\mathrm{E}_t\left[\frac{dP_t^{\$(\tau)}}{P_t^{\$(\tau)}}\right] = i_t dt + a\int_0^T z_t^{\$}(s)\mathrm{cov}_t\left[\frac{dP_t^{\$(\tau)}}{P_t^{\$(\tau)}},\frac{dP_t^{\$(s)}}{P_t^{\$(s)}}\right] + z_t(s)\mathrm{cov}_t\left[\frac{dP_t^{\$(\tau)}}{P_t^{\$(\tau)}},\frac{dP_t^{(s)}}{P_t^{(s)}}\right] ds \quad (47)$$

$$\mathrm{E}_t\left[\frac{dP_t^{(\tau)}}{P_t^{(\tau)}}\right] = (i_t - \pi_t)\, dt + a\int_0^T z_t^{\$}(s)\mathrm{cov}_t\left[\frac{dP_t^{(\tau)}}{P_t^{(\tau)}},\frac{dP_t^{\$(s)}}{P_t^{\$(s)}}\right] + z_t(s)\mathrm{cov}_t\left[\frac{dP_t^{(\tau)}}{P_t^{(\tau)}},\frac{dP_t^{(s)}}{P_t^{(s)}}\right] ds \quad (48)$$

for all τ. Denote log bond prices as $p_t^{\$(\tau)} \equiv \log P_t^{(\$\tau)}$ and $p_t^{(\tau)} \equiv \log P_t^{(\tau)}$. By Itô's Lemma, for any two bond maturities τ and s, we have

$$\mathrm{E}_t\left[dp_t^{(\tau)}\right] = \mathrm{E}_t\left[\frac{dP_t^{(\tau)}}{P_t^{(\tau)}}\right] - \frac{1}{2}\mathrm{var}_t\left[dp_t^{(\tau)}\right] \quad (49)$$

and

$$\mathrm{cov}_t\left[dp_t^{(\tau)}, dp_t^{(s)}\right] = \mathrm{cov}_t\left[\frac{dP_t^{(\tau)}}{P_t^{(\tau)}},\frac{dP_t^{(s)}}{P_t^{(s)}}\right] \quad (50)$$

for real bonds, and similarly for nominal bonds. Making use of these relationships, and substituting in $z^{\$}(\tau) = x^{\$}$ and $z(\tau) = x$ for all τ, we have

$$\mathrm{E}_t\left[dp_t^{\$(\tau)}\right] = i_t dt + a\int_0^T z_t^{\$}(s)\mathrm{cov}_t\left[dp_t^{\$(\tau)}, dp_t^{\$(s)}\right] + z_t(s)\mathrm{cov}_t\left[dp_t^{\$(\tau)}, dp_t^{(s)}\right] ds - \frac{1}{2}\mathrm{var}_t\left[dp_t^{\$(\tau)}\right] \quad (51)$$

$$\mathrm{E}_t\left[dp_t^{(\tau)}\right] = (i_t - \pi_t)dt + a\int_0^T z_t^{\$}(s)\mathrm{cov}_t\left[dp_t^{(\tau)}, dp_t^{\$(s)}\right] + z_t(s)\mathrm{cov}_t\left[dp_t^{(\tau)}, dp_t^{(s)}\right] ds - \frac{1}{2}\mathrm{var}_t\left[dp_t^{(\tau)}\right] \quad (52)$$

Because nominal bonds payoff a nominal face value of 1 with certainty at maturity—that is, they satisfy the boundary condition $p_t^{\$(0)} = 0$ at all t—bonds with positive maturities are given by the sum of expected future returns:

$$p_t^{\$(\tau)} = -\int_0^T \mathrm{E}_t \left[\frac{dp_{t+s}^{\$(\tau-s)}}{ds} \right] ds \tag{53}$$

Real bonds at maturity pay off the rate of inflation in addition to their price appreciation, leading to the solution

$$p_t^{(\tau)} = -\int_0^T \mathrm{E}_t \left[\frac{dp_{t+s}^{(\tau-s)} + \pi_{t+s}}{ds} \right] ds \tag{54}$$

The following standard relationships then determine bond yields $y_t^{(\tau)}$:

$$y_t^{\$(\tau)} \equiv -p_t^{\$(\tau)}/\tau \tag{55}$$

$$y_t^{(\tau)} \equiv -p_t^{(\tau)}/\tau \tag{56}$$

9 Appendix B. Estimated State-Space Model

This appendix describes the empirical model used to obtain the parameters governing the dynamics of the endogenous state variables π, y, and i. The empirical counterparts for these model objects are headline CPI inflation, the CBO output gap, and the three-month Treasury bill rate. However, for a variety of reasons, these data series do not correspond exactly to the model concepts. I therefore assume that the I assume that the data measure the "true" values of inflation, the output gap, and with serially uncorrelated (but possibly cross-correlated) errors. In addition, of course, the shadow rate is not observed at all when it is below the ELB.

From the perspective of the state-space model, the ten-year real yield and the neutral real rate (r*) are exogenous forcing variables. (Of course, in the equilibrium model developed the text, the real yield is made endogenous.). I use the 10-year zero-coupon TIPS yield reported by Gurkaynak et al. (2007) and the difference between long-horizon CPI and T-bill forecasts reported in the Blue Chip Economic Indicators. The latter is a consensus survey forecast at a 6 - 11 year horizon.

Because of the nonlinearity associated with the ELB, analytical filtering methods are not available. The model is estimated using a particle filter. Data are quarterly from 1999:1 - 2021:1. As noted in the text, to aid identification in the relatively short sample, some of the fixed parameters are calibrated to external information. The remaining parameters are estimated by Gibbs sampling, using 100,000 draws with the first 50,000 discarded for burn-in.

Smoothed posterior state estimates are shown in Figure B.1, together with the corresponding data series.

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Table 1. Parameter Values

Description	Parameter	Value	Calibrated to…
Inflation target	π^*	2.4%	Average long-run BC CPI forecast
Inflation inertia	$\exp(-\kappa_\pi)$	0.51	Estimated state-space model*
Inflation response to lag 10Y real yield ι	$\phi_{\pi,y}$	-0.086	Estimated state-space model*
Inflation response to lag GDP gap	$\phi_{\pi,g}$	0.018	Estimated state-space model*
Inflation innovation std. dev.	σ_π	0.37%	Estimated state-space model*
Effective lower bound	b	0%	Assumed zero
Shadow rate inertia	$\exp(-\kappa_i)$	0.76	Carlstrom & Fuerst (2008)
Shadow rate target response to inflation	$\phi_{i,\pi}$	1.5	Taylor (1993)
Shadow rate target response to GDP gap	$\phi_{i,g}$	0.5	Taylor (1993)
Shadow-rate innovation std. dev.	σ_i	0.30%	Estimated state-space model*
Output gap inertia	$\exp(-\kappa_g)$	0.87	Estimated state-space model*
Output gap response to lag 10Y real yield	$\phi_{g,y}$	-0.08	Estimated state-space model*
Output gap response to lag inflation	$\phi_{g,\pi}$	0.17	Estimated state-space model*
Output gap innovation std. dev.	σ_g	0.56%	Estimated state-space model*
Inflation response to output gap innovation	β	0.16	Estimated state-space model*
Risk aversion	a	1	Normalization

* Estimates are posterior means of a nonlinear state-space model over CPI inflation, the Tbill rate, and the output gap. Model is conditioned on the calibrated values of the other parameters. See Appendix B for details.

Table 2. Initial conditions in alternative recessionary interest-rate scenarios

	High rate (Similar to 2008-9)	Moderate rate (Similar to 2020)	Low rate (Hypotehtical)
Inflation (π_0)	1%	1%	1%
Output gap (g_0)	-4%	-4%	-4%
Shadow rate ($\hat{\imath}_0$)	0%	0%	0%
Eq. real short rate (r^*)	1.7%	0%	-1.7%
Nominal bond parameter ($x^\$$)	0.47	-0.03	-0.03
Real bond parameter (x)	-0.69	0.12	0.12
10y nominal yield ($y_0^{\$(40)}$)	4.3%	1.4%	0.0%
10y real yield ($y_0^{(40)}$)	1.6%	-0.6%	-1.0%
QE shock to $x^\$$ or x	-0.055	-0.055	-0.055
Shock to $\hat{\imath}_0$ to "neutralize" $E_0[i]$ under nominal QE	-54%	-341%	0%
Shock to $\hat{\imath}_0$ to "neutralize" $E_0[i]$ under real QE	-21%	-116%	0%

Table 3. Summary of scenario analysis

A. Baseline (feedback neutralized)

	High rate (Similar to 2008-9)		Moderate rate (Similar to 2020)		Low rate (Hypotehtical)	
	Nom. QE shock	Real QE shock	Nom. QE shock	Real QE shock	Nom. QE shock	Real QE shock
Initial effect on yields (bp)						
$y^{\$(40)}$	-100	-68	-87	-54	0	0
$y^{(40)}$	-86	-58	-55	-38	0	-30
10y infl. comp.	-14	-10	-32	-16	0	+30
Dynamic effect on macro variables (bp)						
π_{10}	+29	+18	+24	+15	0	+12
π_{20}	+28	+18	+30	+17	0	+17
g_{10}	+54	+35	+30	+27	0	+22
g_{20}	+60	+37	+63	+36	0	+35

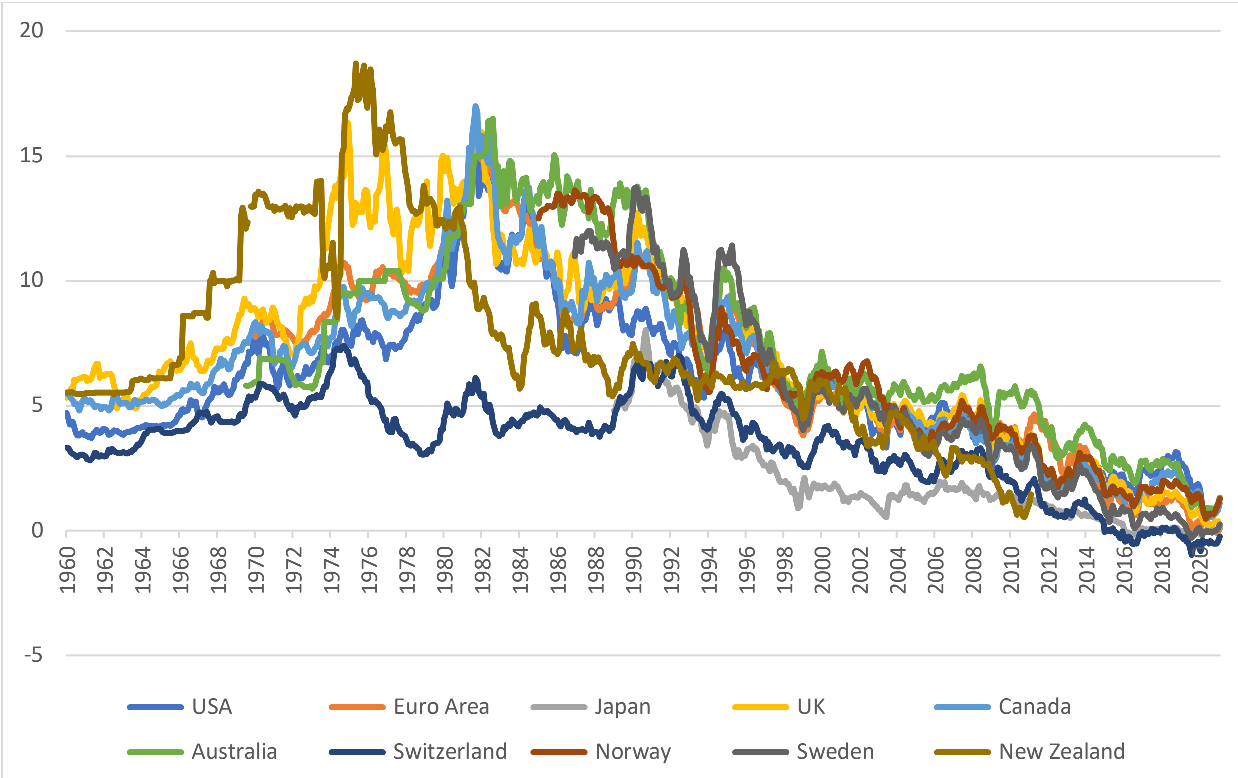
B. Short-rate feedback allowed

	High rate (Similar to 2008-9)		Moderate rate (Similar to 2020)		Low rate (Hypotehtical)	
	Nom. QE shock	Real QE shock	Nom. QE shock	Real QE shock	Nom. QE shock	Real QE shock
Initial effect on yields (bp)						
$y^{\$(40)}$	-71	-46	-62	-33	0	0
$y^{(40)}$	-44	-29	-30	-17	0	-30
10y infl. comp.	-27	-17	-32	-16	0	+30
Dynamic effect on macro variables (bp)						
π_{10}	+16	+11	+13	+7	0	+12
π_{20}	+21	+14	+18	+10	0	+17
g_{10}	+30	+20	+23	+13	0	+22
g_{20}	+43	+28	+36	+21	0	+35

Table 4. Scenario analysis allowing for signaling channel

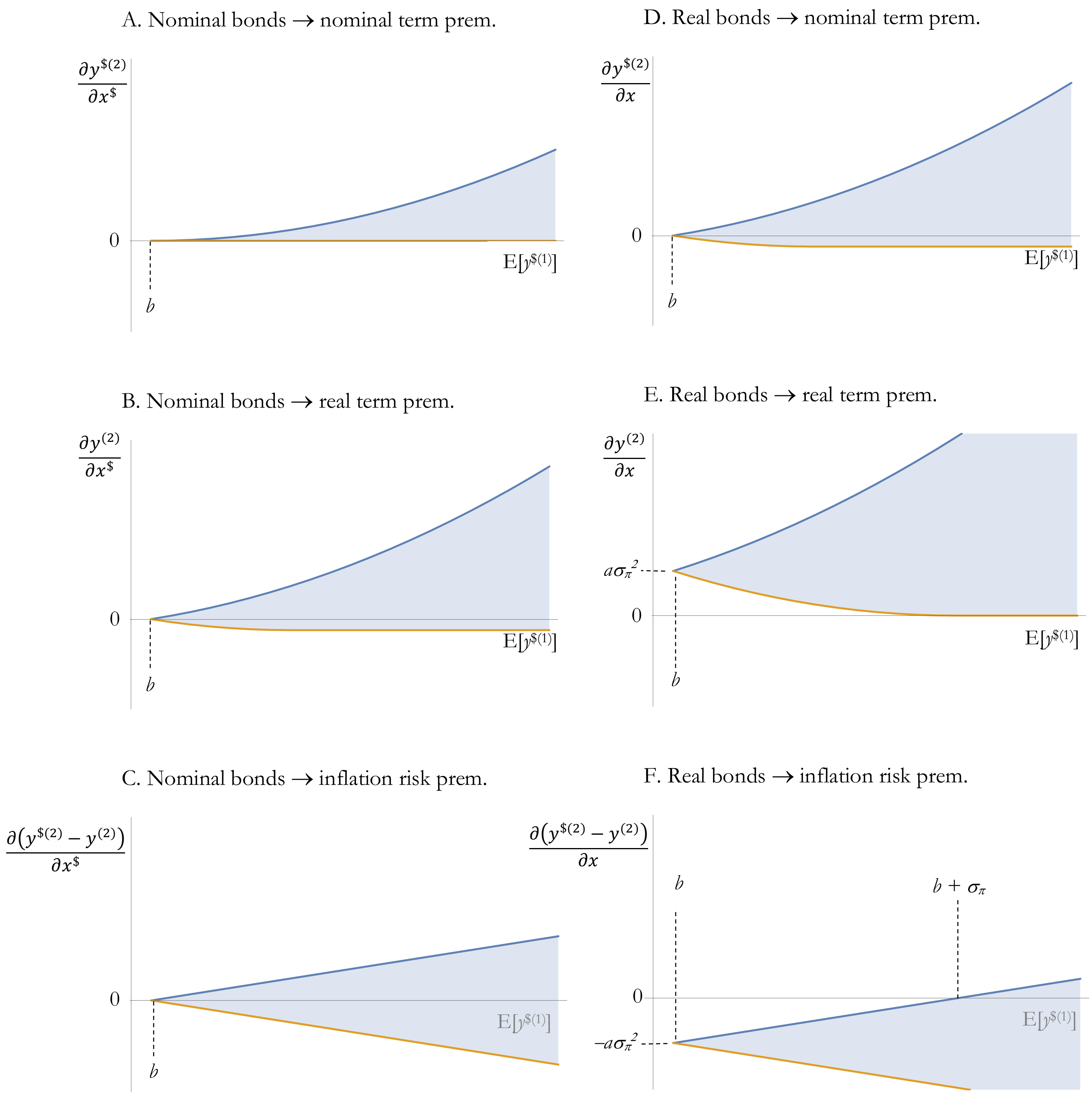
	High rate (Similar to 2008-9)		Moderate rate (Similar to 2020)		Low rate (Hypotehtical)	
	Nom. QE shock	Real QE shock	Nom. QE shock	Real QE shock	Nom. QE shock	Real QE shock
Initial effect on yields (bp)						
$y^{\$(40)}$	-100	-83	-66	-49	0	0
$y^{(40)}$	-105	-95	-43	-38	0	-16
10y infl. comp.	+5	+12	-23	-11	0	+16
Dynamic effect on macro variables (bp)						
π_{10}	+31	+27	+17	+14	0	+6
π_{20}	+28	+23	+21	+16	0	+9
g_{10}	+57	+50	+30	+25	0	+11
g_{20}	+62	+52	+44	+34	0	+19

Figure 1. Long-term nominal bond yields



Source: Federal Reserve Economic Data, Federal Reserve Bank of St. Louis

Figure 2. Possible sensitivities of term premia to bond quantities in 2-period model

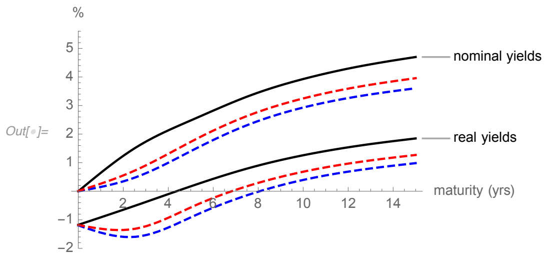


Notes. The figures show the ranges of possible values for the response of nominal and real term premia and the inflation risk premium to changes in the outstanding amounts of real and nominal bonds across values of next period's expected one-period yield in the two-period model.

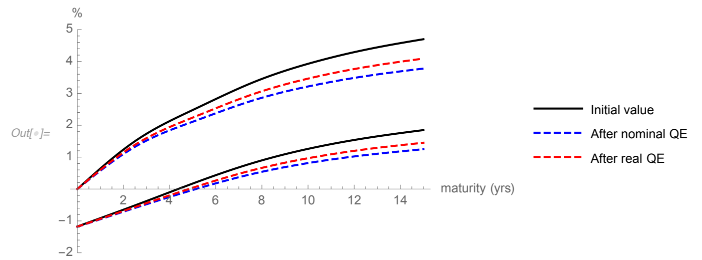
Figure 3. Initial yield-curve responses to QE shocks in the quantitative model

High-rate scenario

A. Baseline (feedback neutralized)

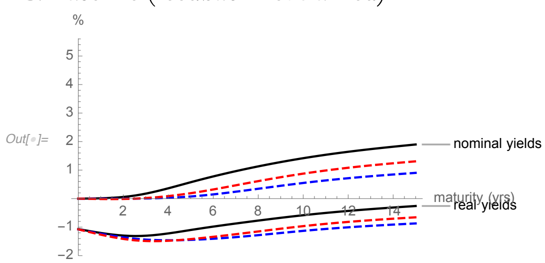


B. Short-rate feedback allowed

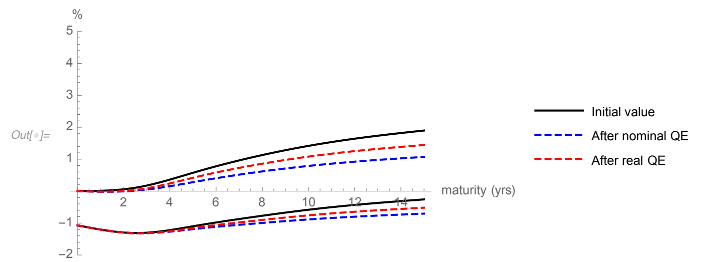


Moderate-rate scenario

C. Baseline (feedback neutralized)

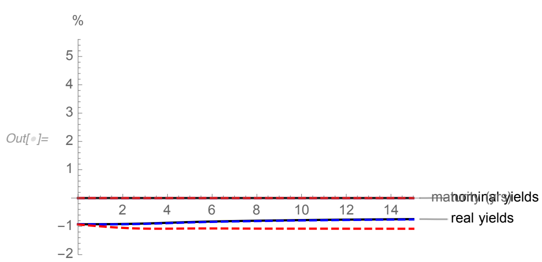


D. Short-rate feedback allowed



Low-rate scenario

E. Baseline (feedback neutralized)



F. Short-rate feedback allowed

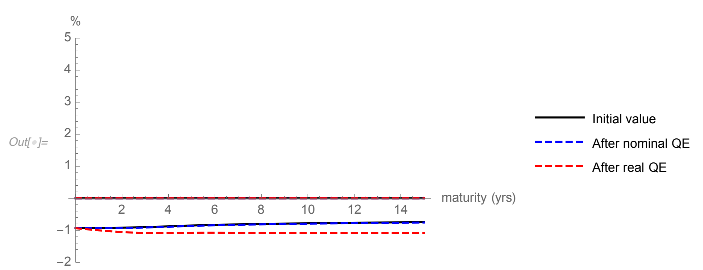
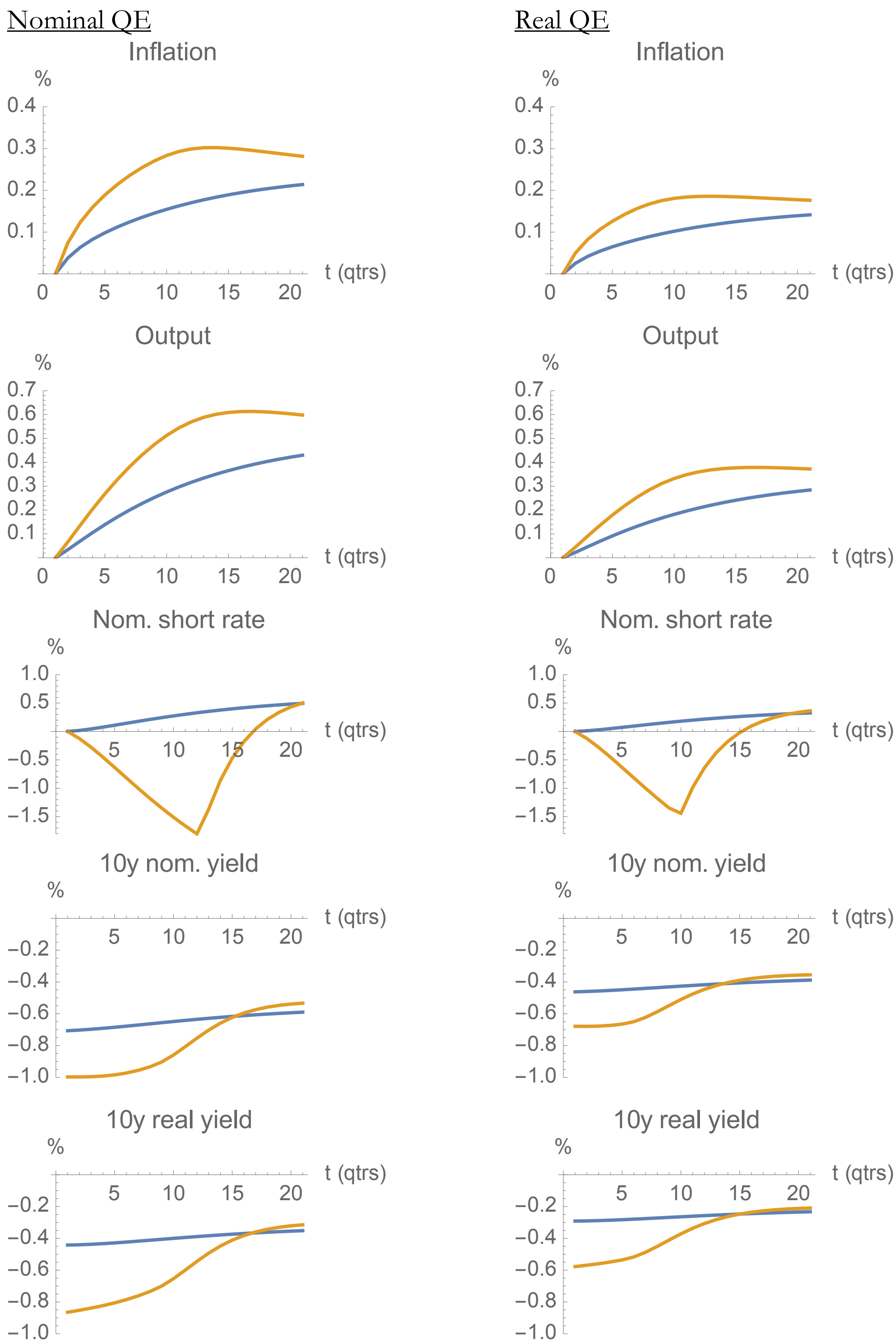
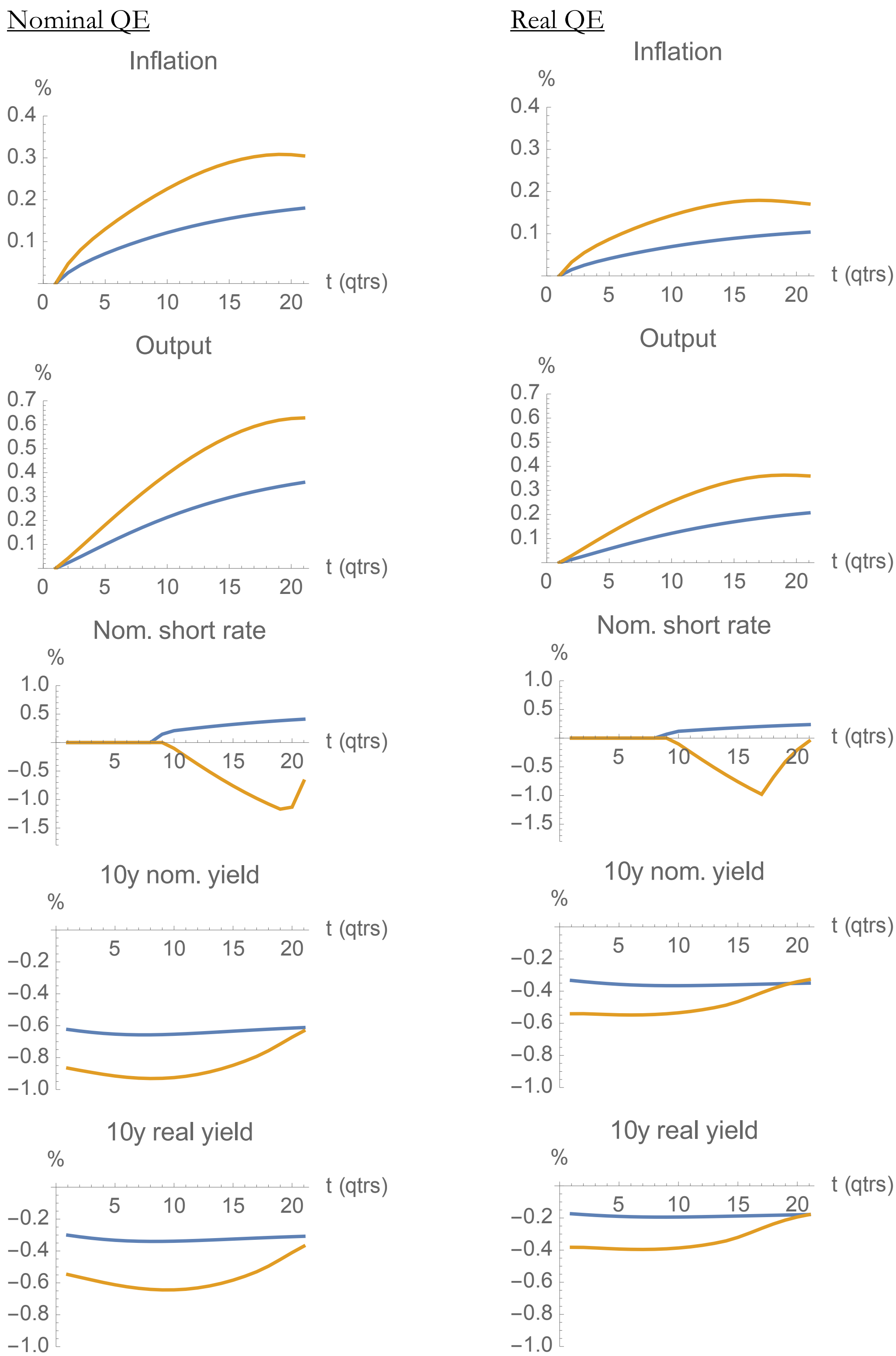


Figure 4. Impulse-response functions in the high-rate scenario



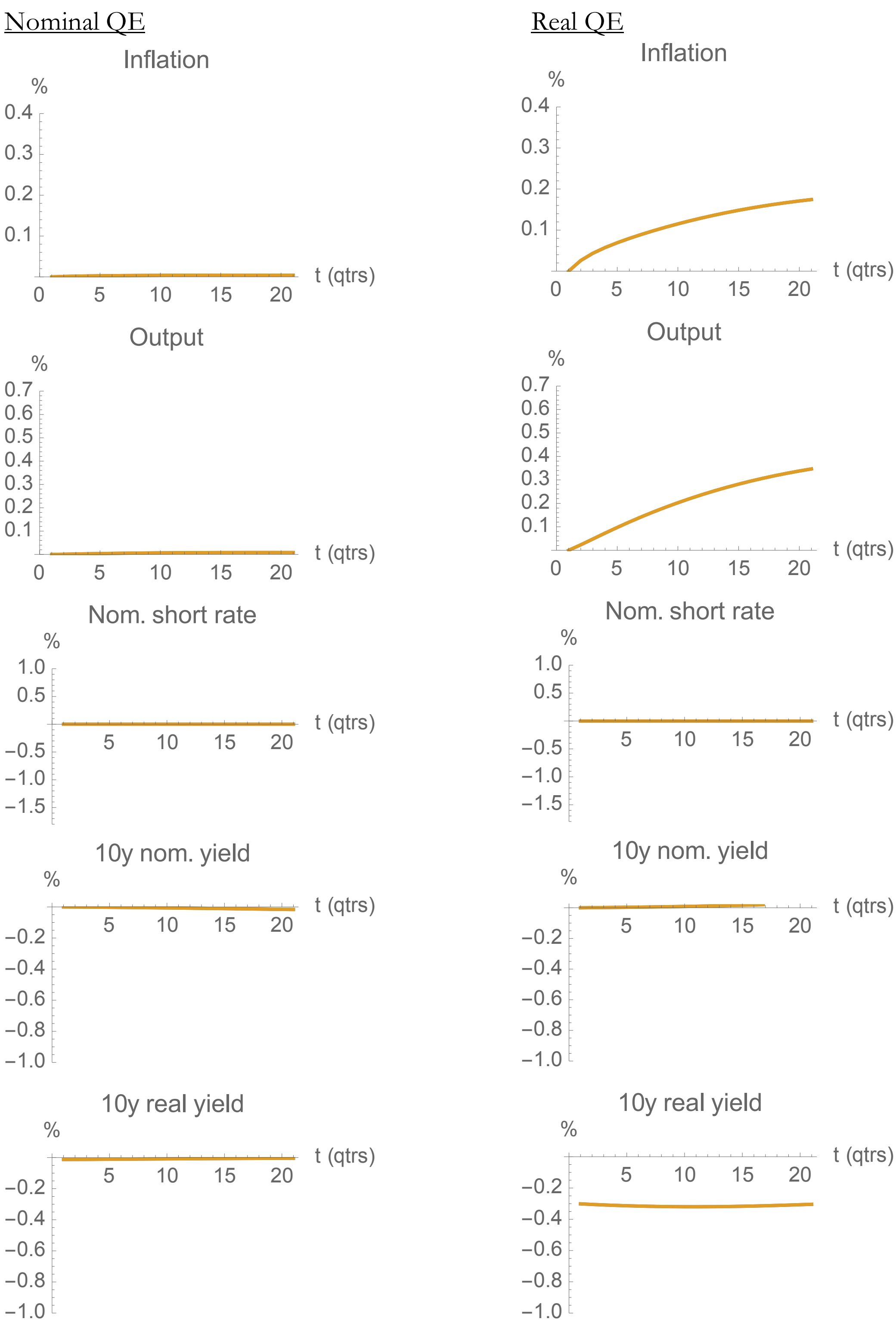
Gold – Baseline (feedback neutralized); Blue – Short-rate feedback allowed

Figure 5. Impulse-response functions in the moderate-rate scenario



Gold – Baseline (feedback neutralized); Blue – Short-rate feedback allowed

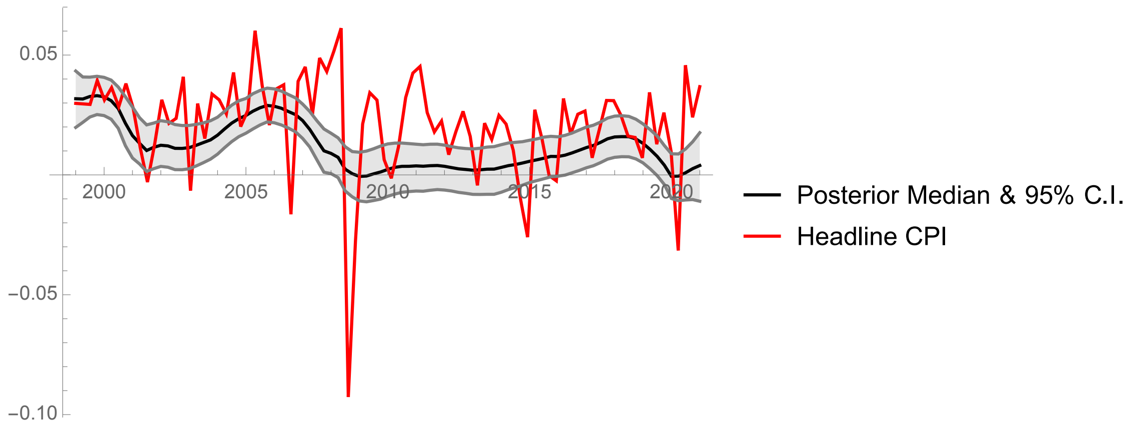
Figure 6. Impulse-response functions in the low-rate scenario



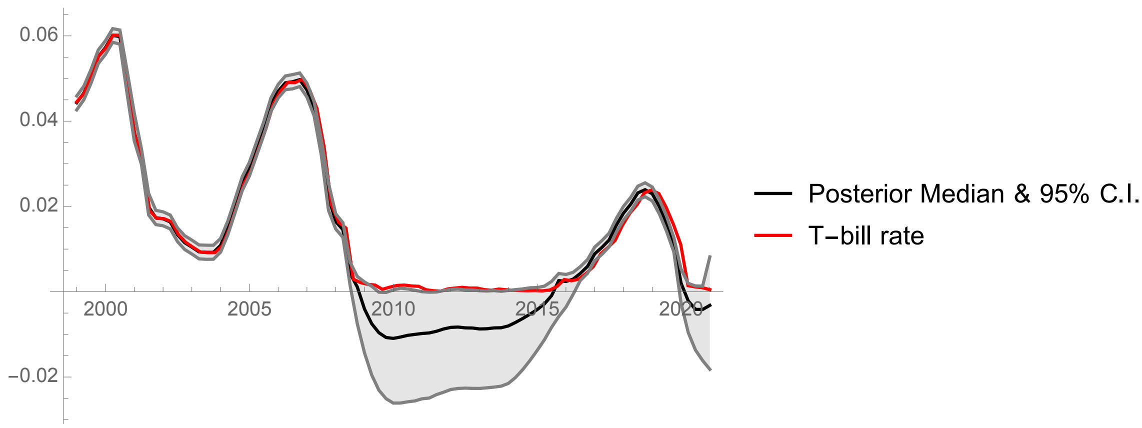
Gold – Baseline (feedback neutralized); Blue – Short-rate feedback allowed

Figure A1. Smoothed state estimates

Inflation



Shadow rate



Output gap